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Weak Field Approach in $f(R)$-Gravity

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Abstract. In this communication we discuss the Weak Field Approach, and in particular the Newtonian limit, applied to $f(R)$-Gravity. Particular emphasis is placed on the spherically symmetric solutions and finally, it is clearly shown that General Relativity results, in the Solar System context, are easily recovered since Einstein theory is a particular case of $f(R)$-Gravity. This is a crucial point against several wrong results in literature stating that these theories are not viable at local scales.

Key words. Alternative theories of Gravity; Newtonian limit; Weak Field limit.

1. Introduction

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration (Perlmutter 2006) has attracted a good amount of interest in Fourth Order Gravity (FOG) considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of Dark Matter and Dark Energy. Other issues come from Astrophysics. For example, the observed Pioneer anomaly problem (Anderson 2002) can be framed into the same approach (Bertolami 2007(@) and then a systematic analysis of such theories urges at small, medium and large scales. Other main topic is the flatness of the rotation curves of spiral galaxies. In particular, a delicate point is to address the weak field limit of any theory of Gravity since two main issues are extremely relevant: *i*) preserving the results of General Relativity (GR) at local scales since they well fit Solar System experiments and observations; *ii*) enclosing in a self-consistent and comprehensive picture phenomena as anomalous acceleration or Dark Matter at Galactic scales.

The idea to extend Einstein's theory of Gravitation is fruitful and economic also with respect to several attempts which try to solve problems by adding new and, most of times, unjustified ingredients in order to give self-consistent pictures of dynamics. Both the issues could be solved by changing the gravitational sector, *i.e.* the left hand side of field equations. In particular, relaxing the hypothesis that gravitational Lagrangian has to be only a linear function of the Ricci curvature scalar R, like in the Hilbert-Einstein formulation, one can take into account an effective action where the gravitational Lagrangian includes a generic function of Ricci scalar ($f(R)$-Gravity).

In this communication, we report the general approach of the Weak Field Limit for $f(R)$-Gravity in the metric approach. We deduce the field equations and derive the weak field potentials with corrections to the Newtonian potential.

2. The Field Equations and their Solutions

Let us start with a general class of $f(R)$-Gravity given by the action

$$\mathcal{A} = \int d^4x \sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m] \tag{1}$$

where f is an unspecified function of curvature invariant R. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. In the metric approach, the field equations are obtained by varying (1) with respect to $g_{\mu\nu}$. We get

$$f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X} T_{\mu\nu} \tag{2}$$

Here, $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the the energy-momentum tensor of matter, while $f' = \frac{df(R)}{dR}$, $\Box = {}_{;\sigma}{}^{;\sigma}$ and $\mathcal{X} = 8\pi G$[1].

The paradigm of Weak Field or Newtonian limit is starting from a develop of the spherically symmetric metric tensor with respect to dimensionless quantity v. To solve the problem we must start with the determination of the metric tensor $g_{\mu\nu}$ at any level of develop (for details see (Capozziello 2007, 2009)). From lowest order of field equations (2) we have $f(0) = 0$ which trivially follows from the assumption that the space-time is asymptotically Minkowskian. A such result suggests a first consideration. If the Lagrangian is developable around a vanishing value of the Ricci scalar we don't have a cosmological constant contribution in the $f(R)$-Gravity.

Let us consider a ball-like source with mass M and radius ξ. The energy-momentum tensor $T_{\mu\nu}$ has the components $T_{tt} \sim T^{(0)}_{tt} = \rho$ and $T_{ij} = T_{0i} = 0$ where ρ is the mass density (we are not interesting to the internal structure). The field equations (2) at $O(2)$ - order become[2]

$$\begin{cases} R^{(2)}_{tt} - \frac{R^{(2)}}{2} + \frac{\triangle R^{(2)}}{3m^2} = \mathcal{X}\rho \\ \\ \frac{\triangle R^{(2)}}{m^2} - R^{(2)} = \mathcal{X}\rho \end{cases} \tag{3}$$

where $\triangle$ is the Laplacian in the flat space, $R^{(2)}_{tt}$ is the time components of Ricci tensor and $m^{-2} = -3f''(0)$. The second line of (3) is the trace of field equations (2) at $O(2)$ - order. It notes that if $f \to R$ (*i.e.* m^2 diverges) the equations (3) correspond to one of GR.

The solution for the Ricci scalar $R^{(2)}$ in the third line of (3) is

$$R^{(2)}(t, \mathbf{x}) = m^2\mathcal{X}\int d^3\mathbf{x}'\mathcal{G}(\mathbf{x}, \mathbf{x}')\rho(t, \mathbf{x}') \tag{4}$$

where $\mathcal{G}(\mathbf{x}, \mathbf{x}')$ is the Green function of field operator $\triangle - m^2$. The solution for $g^{(2)}_{tt}$, from the first line of (3) by considering that $R^{(2)}_{tt} = \frac{1}{2}\triangle g^{(2)}_{tt}$, is

$$\begin{aligned} g^{(2)}_{tt}(t, \mathbf{x}) = &-\frac{\mathcal{X}}{2\pi}\int d^3\mathbf{x}'\frac{\rho(t, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &-\frac{1}{4\pi}\int d^3\mathbf{x}'\frac{R^{(2)}(t, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{2}{3m^2}R^{(2)}(t, \mathbf{x}) \end{aligned} \tag{5}$$

We can check immediately that when $f \to R$ we find $g^{(2)}_{tt}(t, \mathbf{x}) \to -2G\int d^3\mathbf{x}'\frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$ (Stabile 2010). The solution (5) is the the gravitational potential $\Phi = g^{(2)}_{tt}/2$ for $f(R)$-Gravity. We note that Φ has a Yukawa-like behavior depending by a characteristic length on which it evolves. As it is evident the Gauss theorem is not valid since the force law is not $\propto |\mathbf{x}|^{-2}$. The equivalence between a spherically symmetric distribution and point-like distribution is not valid and how the matter is distributed in the space is very important (Capozziello 2009).

[1] Here we use the convention $c = 1$.

[2] We set for simplicity $f'(0) = 1$ (otherwise we have to renormalize the coupling constant $\mathcal{X}$ in the action (1)).

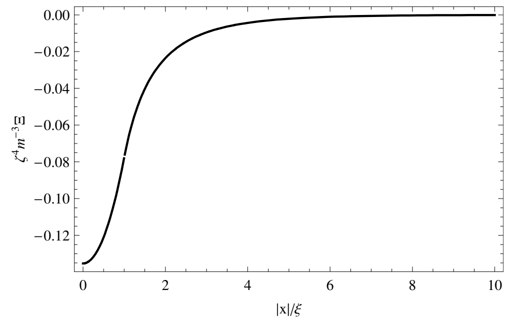

Fig. 1. Plot of dimensionless function $\zeta^4\mu^{-3}r_g{}^{-1}R^{(2)}$ for $\zeta = \mu\xi = .5$ representing the spatial behavior of Ricci scalar at second order.

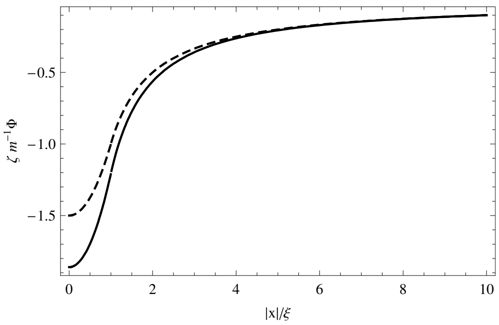

Fig. 2. Plot of metric potential $2\zeta\mu^{-1}r_g{}^{-1}\Phi$ vs distance from central mass with $\zeta = \mu\xi = .5$. The dashed line is the GR behavior.

From the solution (5) we can affirm that it is possible to have solutions non-Ricci-flat in vacuum: *Higher Order Gravity mimics a matter source*. It is evident from (5) the Ricci scalar is a "matter source" which can curve the spacetime also in absence of ordinary matter. Besides the solutions are depending on the only first two derivatives of f in $R = 0$. So different theories from the third derivative admit the same solutions.

3. The Spatial Behaviors of Gravitational Potential

If $m^2 > 0$ we have as Green function with spherical symmetry the following expression

$$\mathcal{G}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{6}$$

where we defined $\mu = \sqrt{|m^2|}$. Then the spatial behaviors of Ricci scalar (4) and gravitational potential Φ, if ρ = constant, are shown in the Figs. 1 and 2.

For fixed values of the distance $|\mathbf{x}|$, the solution $g_{tt}^{(2)}$ depends on the value of the radius ξ, then the Gauss theorem does not work also if the Bianchi identities hold. We can affirm: *the potential does not depend only on the total mass but also on the mass - distribution in the space*.

It is interesting to note as the gravitational potential assumes smaller value of its equivalent in GR, then in terms of gravitational attraction we have a potential well more deep. Besides if the mass distribution takes a bigger volume, the potential increases and vice versa.

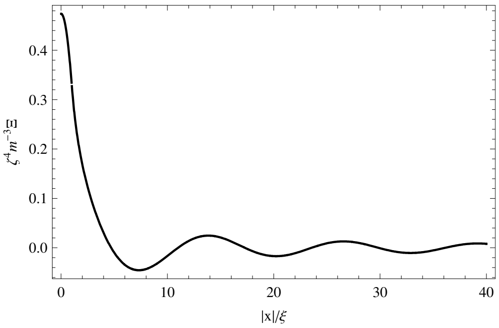

Fig. 3. Plot of dimensionless function $\zeta^4\mu^{-3}r_g{}^{-1}R^{(2)}$ with $\zeta = \mu\xi = .5$ representing the spatial behavior of Ricci scalar at second order in the oscillating case.

If $m^2 < 0$ the Green function assumes the "oscillating" expression

$$\mathcal{G}(\mathbf{x},\mathbf{x}') = -\frac{\cos\mu|\mathbf{x}-\mathbf{x}'| + \sin\mu|\mathbf{x}-\mathbf{x}'|}{4\pi\,|\mathbf{x}-\mathbf{x}'|} \tag{7}$$

Now the Ricci scalar (4) and gravitational potential Φ are shown in Figs. 3 and 4.

Finally in the limit of point-like source, *i.e.* $\rho = M\,\delta(\mathbf{x})$, we get

$$\begin{cases} R^{(2)} = -r_g\mu^2\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} \\ \Phi = -\frac{r_g}{2}\left(\frac{1}{|\mathbf{x}|} + \frac{1}{3}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}\right) \end{cases} \tag{8}$$

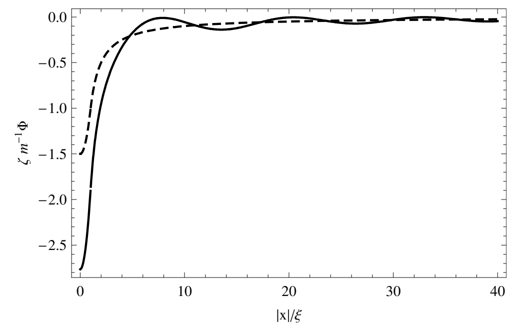

Fig. 4. Plot of metric potential $2\zeta\mu^{-1}r_g{}^{-1}\Phi$ vs distance from central mass with the choice $\zeta = \mu\xi = .5$ in the oscillating case. The dashed line is the GR behavior.

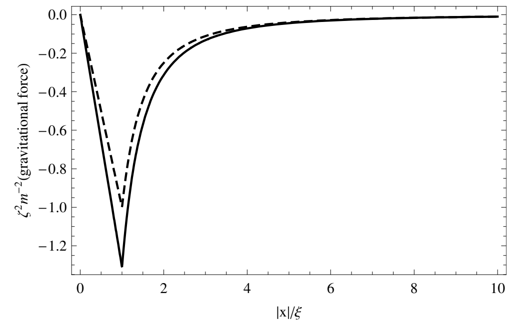

Fig. 5. Comparison between gravitational forces induced by GR and $f(R)$-Gravity with $\zeta = \mu\xi = .5$. The dashed line is the GR behavior.

where $r_g = 2GM$ is the Schwarzschild radius. If $f(R) \to R$ we recover the gravitational potential induced by GR.

To conclude this section we show in Fig. 5 the comparison between gravitational forces induced in GR and in $f(R)$-Gravity in the Newtonian limit. Obviously also about the force we obtained an intensity stronger than in GR.

4. Conclusions

The Weak Field Limit is a crucial issue that has to be addressed in any relativistic theory of Gravity. It is also the test bed of such theories in order to compare them with the well-founded experimental results of GR, at least at Solar system level.

The general feature that emerges from the Weak Field Limit is that correction to the Newtonian potential naturally comes out. This correction is a Yukawa-like term bringing characteristic mass and length. Conversely, the standard Newtonian potential is just a feature emerging in the particular case $f(R) = R$.

It is well-known that the new features related to FOG could have interesting applications in other fields of Astrophysics as galactic dynamics, large scale structure and Cosmology in order to address Dark Matter and Dark Energy issues. The fact that such "dark" structures have not been definitely discovered at fundamental quantum scales but operate at large astrophysical (infra-red scales) could be due to these corrections to the Newtonian potential which can be hardly detected at laboratory or Solar System scales.

Finally, the presence of unavoidable light massive modes could open new opportunities also for the gravitational waves detection of experiments like VIRGO, LIGO and the forthcoming LISA.

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