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Smoothed Particle Magnetohydrodynamics

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Abstract. In this paper we describe a recent algorithm for the simulation of Magnetohydrodynamics (MHD) using the Smoothed Particle Hydrodynamics (SPH) method. The algorithm is shown to give robust and accurate results on standard test problems used to benchmark recent grid-based MHD codes in one and two dimensions. The new method provides an extremely useful tool for the simulation of the complex astrophysical phenomena associated with magnetic fields, such as those involved in the star formation process.

Key words. methods: numerical – magnetohydrodynamics

1. Introduction

Smoothed Particle Hydrodynamics (SPH) is a Lagrangian particle method for solving the equations of fluid dynamics (for a review see Monaghan 1992). Rather than using a spatial grid, fluid quantities and their derivatives are evaluated on a set of moving interpolation points which follow the fluid (the 'particles'). SPH has many advantages, particularly in astrophysical simulations, as it imposes no restrictions on the symmetry of the problem to be solved and therefore handles complex physics with relative ease. Since each particle is associated with a fixed amount of mass, the spatial resolution automatically changes according to the density distribution, removing the need for complicated mesh-refinement procedures. The discrete equations of motion used in SPH can be derived self-consistently from a variational principle, a fact that can be exploited to derive consistent formulations of the SPH equations which lead to increased accuracy (Nelson & Papaloizou 1994; Monaghan & Price 2001; Springel & Hernquist 2002; Price & Monaghan 2003b). In particular this can be useful in situations where the physics is somewhat complicated, such as in general relativity (Monaghan & Price 2001) and in the present case of magnetohydrodynamics (Price & Monaghan 2003b), as we will discuss in this paper.

The inclusion of magnetic fields was considered in one of the first SPH papers (Gingold & Monaghan 1977), with application to magnetic polytropes. A more detailed investigation (Phillips & Monaghan 1985) revealed a numerical instability that occurred when an exactly momentum-conserving form of the discrete equations were used. This instability took the form of particles tending to clump together under negative (tensile) stresses (Morris 1996). Despite the presence of this instability, a good number of SPH simulations which include magnetic fields have been performed since other formulations could be used which either do not require the exact conservation of momentum (e.g. Benz 1984; Meglicki 1995;

Cerqueira & de Gouveia Dal Pino 2001), or which use the momentum-conserving form in the regime where the instability does not occur (e.g. Dolag et al. 1999; Marinho et al. 2001). The advantage of using an exactly momentum-conserving formulation is that it gives much better results for shock-type problems. In the MHD case there are other advantages relating to a consistent treatment of terms proportional to the divergence of the magnetic field, as we discuss in §3.

We have recently presented an algorithm for Smoothed Particle Magnetohydrodynamics (SPMHD) which does not suffer from instabilities (Price & Monaghan 2003a,b, hereafter papers I, II) by implementing a fix for the tensile instability proposed in the context of elastic dynamics problems by Monaghan (2000). The discrete equations are formulated from a variational principle which ensures consistency with physical principles (such as conservation of momentum and energy) and a consistent treatment of magnetic divergence terms. We also formulate artificial dissipation terms appropriate for shock-type problems. These terms are carefully formulated to give a positive definite contribution to the entropy. The algorithm has been tested on a wide range of standard one and two dimensional problems used to test recent grid-based MHD codes and is demonstrated to give robust and accurate results.

In this paper we briefly describe our SPMHD algorithm and present the results of several of these test problems.

2. Magnetohydrodynamics

The equations of fluid dynamics in the magnetohydrodynamics (MHD) approximation consist of the continuity equation,

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}, \tag{1}$$

the momentum equation,

$$\frac{dv^i}{dt} = \frac{1}{\rho}\frac{\partial S^{ij}}{\partial x^j}, \tag{2}$$

where

$$S^{ij} = -P\delta^{ij} + \frac{1}{\mu_0}\left(B^iB^j - \frac{1}{2}\delta^{ij}B^2\right), \tag{3}$$

together with an appropriate energy equation, which for the total specific energy $\widehat{e} = \frac{1}{2}v^2 + u + \frac{1}{2}B^2/(\mu_0\rho)$ is given by

$$\frac{d\widehat{e}}{dt} = \frac{1}{\rho}\frac{\partial(v^iS^{ij})}{\partial x^j}, \tag{4}$$

The magnetic field $\mathbf{B}$ is updated according to the induction equation,

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \tag{5}$$

where η is the magnetic diffusivity ($\eta = 0$ in 'ideal' MHD, that is where infinite conductivity is assumed). The equation set is closed by an appropriate equation of state, which for an ideal gas is given by

$$P = (\gamma - 1)\rho u, \tag{6}$$

where P is the pressure, u represents the internal energy per unit mass and γ is the ratio of specific heats.

3. Divergence terms

When the MHD equations are solved numerically there is an arbitrariness relating to terms proportional to the divergence of the magnetic field, $\nabla\cdot\mathbf{B}$. In the continuum case these terms are zero (by way of the Maxwell equation) and so are not present, however this cannot be true in a discrete approximation due to round off error (this applies even to constrained transport type schemes, since a zero divergence in one discrete approximation does not necessarily imply that it is zero in another - see Tóth 2000). An example of the issue relating to the divergence terms, consider the magnetic force, which when written in the form

$$F_{mag} = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_0\rho} + \frac{\mathbf{B}\nabla\cdot\mathbf{B}}{\mu_0\rho}, \tag{7}$$

may be expressed as the gradient of a symmetric tensor, as in (2). Discrete approximations based on this form of the magnetic force can be constructed so as to conserve momentum exactly. The same is not true if the divergence term is neglected (that is using simply the $\mathbf{J}\times\mathbf{B}$ force, where $\mathbf{J} = \nabla\times\mathbf{B}$ is the current density), although the two are equivalent in the continuum case. A similar issue arises with the induction equation. In 'conservative' form, the induction equation is given by (5) and results in exact conservation of the quantity $\int \mathbf{B}\mathrm{dV}$. Taking the divergence of (5), we find

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) = 0, \tag{8}$$

which demonstrates the manner in which the divergence constraint enters the MHD equations (ie. as an initial condition). Rewriting (5) using the Lagrangian time derivative d/dt gives

$$\frac{d\mathbf{B}}{dt} = -\mathbf{B}(\nabla\cdot\mathbf{v}) + (\mathbf{B}\cdot\nabla)\mathbf{v} + \mathbf{v}(\nabla\cdot\mathbf{B}). \tag{9}$$

Non-zero divergence terms will, however, always arise in a numerical simulation due to round-off error. Allowing magnetic monopoles resulting from $\nabla\cdot\mathbf{B}\neq 0$ to evolve appropriately within the flow can prevent the build up of unphysical numerical effects associated with their presence and can therefore reduce the need for computationally expensive divergence cleaning procedures. Thus Powell et al. (1999) suggested that the conservative forms of the MHD equations should contain source terms (that is, they use non-conservative forms) to ensure that these errors are propagated out by the flow. Neglecting the $\nabla\cdot\mathbf{B}$ term in (9), we find that any spurious magnetic monopoles evolve according to

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) + \nabla\cdot(\mathbf{v}\nabla\cdot\mathbf{B}) = 0, \tag{10}$$

which has a similar form to the continuity equation for the density. In this 'non-conservative' form, the induction equation can be expressed as

$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{v}. \tag{11}$$

However, Powell et al. (1999) also use non-conservative forms of the energy and momentum equations and correspondingly is has been shown by Tóth (2000) that this can lead to some errors at shock fronts. More recently it has been shown by Janhunen (2000) and Dellar (2001) that the correct formulation of the MHD equations in the presence of monopoles should *not* violate the conservation of momentum and energy.

In the SPMHD case, the equations of motion and energy can be derived from a variational principle which uses the discrete forms of the continuity and induction equations as constraints (Price & Monaghan 2003b). In this way we are able to verify that the induction equation (9) with the $\nabla\cdot\mathbf{B}$ term neglected is in fact consistent with a conservative formulation of the magnetic force. We are therefore able to retain the conservation of momentum (resulting in good shock-capturing ability) whilst allowing the monopoles to evolve (helping to control the divergence error). Retaining the exact conservation of momentum in SPMHD requires a solution to the numerical instability, which we discuss below.

4. Smoothed Particle Magnetohydrodynamics

The discrete form of the SPMHD equations are given by the density summation

$$\rho_a = \sum_b m_b W(|\mathbf{r}_a - \mathbf{r}_b|, h), \tag{12}$$

where $W(|\mathbf{r}_a - \mathbf{r}_b|, h)$ is the interpolation kernel with smoothing length h, for which we use the usual cubic spline (Monaghan 1992). This expresses the continuity equation in SPH. The momentum equation is given by

$$\frac{dv_a^i}{dt} = \sum_b m_b \left[\left(\frac{S^{ij}+R^{ij}}{\rho^2}\right)_a + \left(\frac{S^{ij}+R^{ij}}{\rho^2}\right)_b\right]\nabla_a^j W_{ab}, \tag{13}$$

where we have added an artificial viscosity term Π_{ab} in order to handle shocks (discussed in paper I), and we add an additional term to the stress to counteract the numerical instability discussed in §1. This term is proportional to the anisotropic magnetic stress and is given by

$$R^{ij} = -RB^iB^j, \tag{14}$$

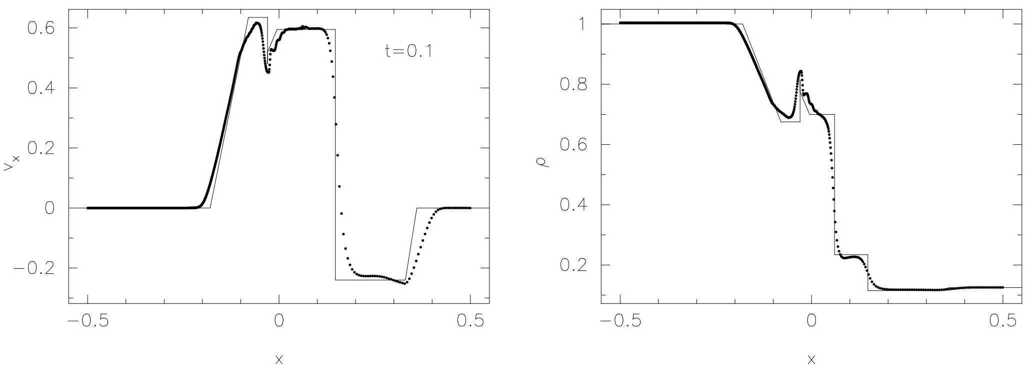

Fig. 1. Results of the Brio & Wu (1988) shock tube test. To the left of the origin the initial state is $(\rho, P, v_x, v_y, B_y) = [1, 1, 0, 0, 1]$ whilst to the right the initial state is $(\rho, P, v_x, v_y, B_y) = [0.125, 0.1, 0, 0, -1]$ with $B_x = 0.75$ everywhere and $\gamma = 2.0$. The velocity (left) and density (right) profiles are shown at time $t = 0.1$ and may be compared with the exact solution given by the solid line. The complexity of the shock in MHD is due to the different types of characteristic waves. The solid points represent the particles.

where R is a function which responds when particles begin to clump together, given by

$$R = \epsilon \left(\frac{W_{ab}}{W(\Delta p)} \right)^n , \tag{15}$$

where W is the SPH kernel and W(Δp) is the kernel evaluated at the average particle spacing and we typically use $\epsilon \sim 0.8$ and $n \sim 5$. We find that such a term very effectively removes the instability with few side effects. This term is preferable to a modification of the smoothing kernel since it does not affect non-MHD flows.

The energy equation (4) in discrete form is given by

$$\begin{aligned} \frac{d\widehat{e}_a}{dt} &= \sum_b m_b \left(\frac{v_a^i S_b^{ij}}{\rho_b^2} + \frac{v_b^i S_a^{ij}}{\rho_a^2} + \Omega_{ab} \right) \frac{\partial W_{ab}}{\partial x_a^j} \\ &+ \sum_b m_b v_a^i \left(\frac{R_a^{ij}}{\rho_a^2} + \frac{R_b^{ij}}{\rho_b^2} \right) \frac{\partial W_{ab}}{\partial x_a^j}, \end{aligned} \tag{16}$$

where Ω_{ab} is a dissipation term analogous to Π_{ab}.

The induction equation (11) is given by

$$\frac{d}{dt} \left(\frac{B^i}{\rho} \right)_a = -\frac{1}{\rho_a^2} \sum_b m_b v_{ab}^i B_a^j \frac{\partial W_{ab}}{\partial x_a^j}, \tag{17}$$

where in practice we also add a dissipative term which is used in shocks.

In practice the smoothing length (which determines the size of the interpolation region for each particle) is evolved according to the local density. Further improvements in accuracy can be made by re-deriving the discrete equations from a variational principle and taking account of this dependence. This is derived and demonstrated in paper II and we use this formulation in the two dimensional tests.

5. Numerical tests in one dimension

We have extensively tested the algorithm on a wide range of one dimensional problems, including both shocks and small amplitude MHD waves (paper I,II). We give a representative example from the shock tube tests here. This test has been used to benchmark nearly all recent grid-based MHD codes and is a magnetic generalisation of the standard Sod (1978) shock tube test for hydrodynamics. The problem consists of two initial states (to the left and right of the origin) brought into contact at $t = 0$. As time progresses various shock structures develop, which appear substantially different in MHD because there are three different types of waves as opposed to just one in the hydrodynamical case. This is illustrated in figure 1 which shows the results of this test at time $t = 0.1$, which may be compared with the ex-

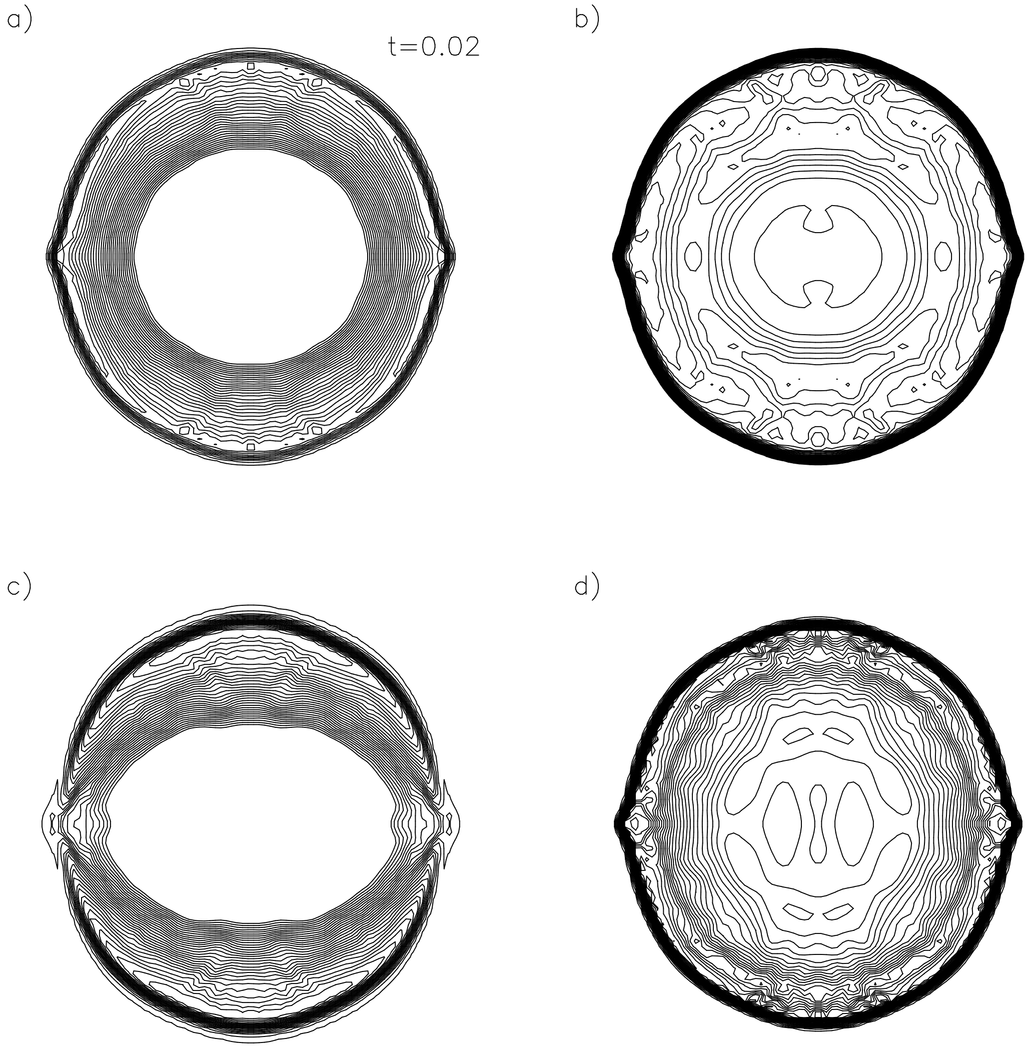

Fig. 2. Spherical adiabatic MHD blast wave in two dimensions. Plots show: a) logarithm to base 10 of the density; b) logarithm to base 10 of the pressure; c) logarithm to base 10 of the magnetic pressure; d) specific kinetic energy. All plots show 30 contours spaced evenly between the minimum and maximum values of the quantity shown. Results compare well with those shown in Balsara (1998)

act solution given by the solid line. Further one dimensional tests are given in papers I and II.

6. Numerical tests in two dimensions

We also present preliminary results from test calculations in two dimensions. The tests shown here are frequently used to benchmark grid-based MHD codes. The first test involves an adiabatic blast wave propagating in a magnetic medium. Initially the pressure is set to 1000 in a spherical region of radius $r = 0.05$ around the origin in a uniform density box containing 100x100 particles. The density is initially unity and in the simulation shown we use

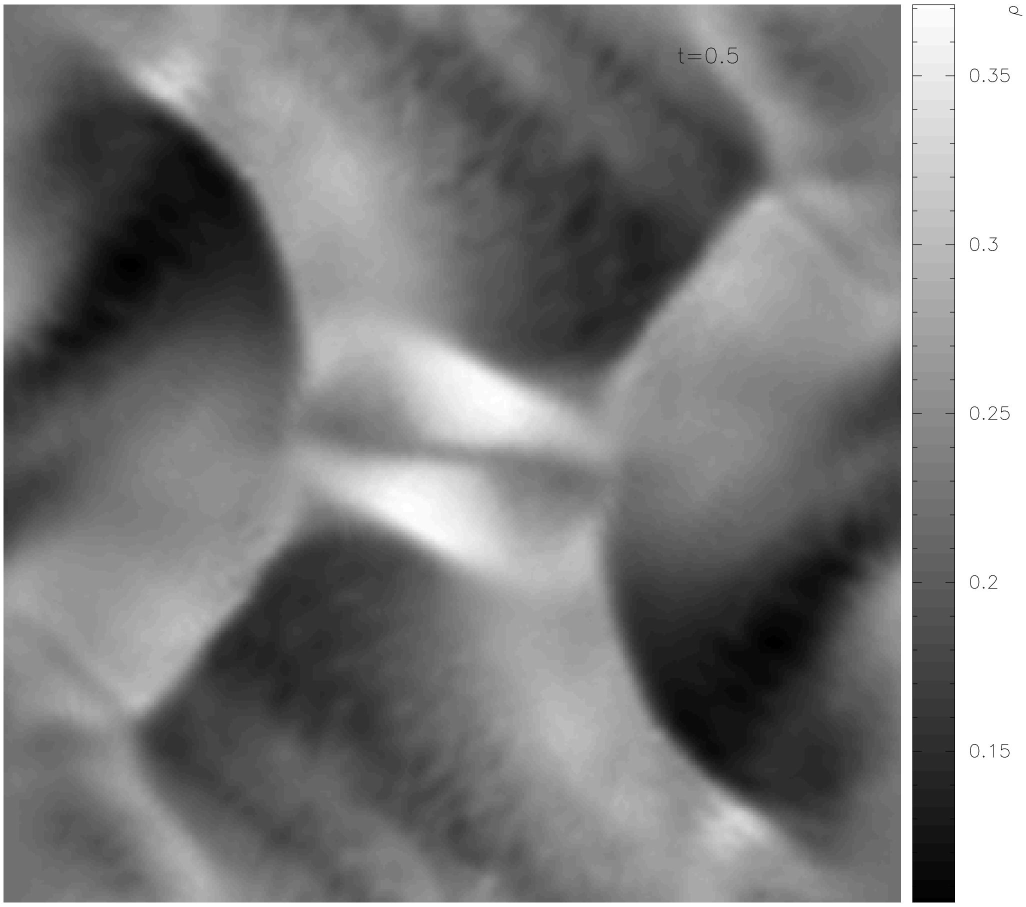

Fig. 3. Results of the two dimensional Orszag-Tang vortex test, showing the density distribution at $t = 0.5$. The simulation uses 16384 particles initially arranged on a cubic lattice, with periodic boundary conditions. As time progresses shocks form and gradually the flow becomes turbulent. The results shown here are in good agreement with those produced by grid-based MHD codes (e.g. Dai & Woodward 1994,1998; Tóth, 2000).

$\gamma = 1.4$. A constant, uniform field of strength 10G (in code units $B_x = 10/\sqrt{4\pi}$) is setup in the x-direction. Results are shown in figure 2 at $t = 0.02$ and compare extremely well to similar results given in Balsara (1998). In particular the contours of density and pressure show very little scatter. This is due to our use of the total energy equation (16), rather than the thermal energy equation as is usual practice in SPH. The advantages of exact energy conservation are clear in this type of strong shock problem. There are some small effects visible due to the regularity of the initial particle setup.

The second two dimensional test has been used to test many grid-based MHD codes. The problem was initially studied by Orszag & Tang (1979) and consists of an initially uniform density, periodic 1x1 box given an initial velocity perturbation $\mathbf{v} = [-\sin(2\pi y), \sin(2\pi x)]$. The magnetic field is given a doubly periodic geometry $\mathbf{B} = [-\sin(2\pi y), \sin(4\pi x)]$. The flow has an initial average Mach number of unity and a ratio of magnetic to thermal pressure of 10/3. We set up 16384 (128x128) particles distributed uniformly on a cubic lattice. As time progresses shocks form and the flow becomes turbulent. The results of the density evolution are shown in figure 3 at $t = 0.5$. The results are in good agreement with those presented elsewhere (e.g. Dai & Woodward 1994, 1998; Tóth 2000), although it is our intention to investigate the code performance on this and other multidimensional test problems in more detail in a further paper.

7. Conclusions

We have briefly described our algorithm for Smoothed Particle Magnetohydrodynamics. The algorithm is shown to give robust and accurate results on a wide range of one and two dimensional problems used to benchmark grid-based MHD codes. On this basis we believe the algorithm will provide an extremely useful tool for the simulation of the complex astrophysical phenomena associated with magnetic fields, such as those involved in the star formation process.

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