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Angular momentum transport in CV accretion disks

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Abstract. A physical mechanism for enhanced angular momentum transport (AMT) outwards is needed for the theoretical modeling of accretion disks (ADs) in e.g. Cataclysmic Variables (CVs). It is clear that ordinary microscopic viscosity is out of the question - the time scale of transport it dictates is many orders of magnitude longer than the one deduced from mass accretion rate consistent with observations. It is however reasonable that *turbulent transport* can provide the needed timescale value. Astrophysicists have persistently looked for an linear instability in ADs, which can ultimately lead to turbulence. In this contribution we dispute the "common knowledge", accepted during the last two decades or so, that the magneto-rotational instability (MRI) is the physical agent that destabilizes thin ADs and ultimately drives turbulence and AMT. We also suggest, thus, that we do not yet have at our disposal any better model for AMT than the30 year old α model

Key words. Binary stars: accretion disks – MHD instabilities: MRI – Transport: angular momentum

1. Introduction

Prendergast & Burbidge (1968) were the first to propose the existence of thin ADs, to explain some galactic X-ray sources. Similar objects were also proposed in CVs. Greatly enhanced (turbulent) viscosity was invoked for making the model viable. A few years later Shakura & Sunayev (1973) and Lynden-Bell & Pringle (1974) successfully bypassed the specific lack of understanding of turbulence (a situation that is largely still with us) by employing physically motivated parametrization of turbulent viscosity. In doing so, thin AD, for the first time, could be quantitatively modeled in a variety of astrophysical settings. In turn, this has brought about significant progress in our understanding of these important systems. The value and nature of the non-dimensional viscosity parameter α (see below) still remains to be a theoretical unknown and phenomenological approaches have usually been employed to determine it.

No definitive candidate hydrodynamical instability had been convincingly identified in ADs and widely accepted as a possible initial driver of turbulence before (and af-

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ter...) Balbus & Hawley (1991) found that weak magnetic fields destabilize differential Keplerian rotation. The linear MRI (Velikhov 1959, Chandrasekhar 1960), was shown to operate under conditions characterizing rotationally supported magnetized "cylindrical accretion disks".

This suggestion has inspired intensive research activity on the linear MRI and its nonlinear development. It has been investigated for various conditions, geometries, and boundary conditions (BC). Beyond the linear analysis it has been approached, almost exclusively, by methods of computational (magneto)fluid dynamics. The complexity of the nonlinear development became readily apparent quickly apparent and, as a consequence some understanding of the *nonlinear* transition and saturation processes, is indispensable, if the aim is to improve on the more than 30 years old phenomenological α-model.

The purpose of this contribution is to assess the viability of the most common approximation used in numerical studies - the *shearing box* (SB). The review articles by Balbus & Hawley (1998) and Balbus (2003) contain a comprehensive list of references, of which only few will be referred to here. We shall also present, in the last section, a semi-analytical study of the saturation of the MRI, in an experimental simplified setup, which has been used in experimental efforts to observe the MRI and its nonlinear development in the laboratory (e.g. Liu, Goodman & Ji 2006 and references therein).

2. Numerical calculations of the MRI nonlinear development

Most of the existing SB calculations (at least of the early ones) employ a base state with a constant vertical magnetic field and numerically follow the development of perturbations on this state. This situation, usually referred to as *fixed net flux* (FF), seems to be the most appropriate for disks threaded by external magnetic fields . The majority of existing numerical works of this sort have used *ideal* (that is, non-dissipative) MHD SB equations with *periodic boundary conditions* (PBC) in the manner employed, for example, by Hawley et al. (1995) (H95), but the very recent calculations of Lesur & Longaretti (2007) included explicit viscosity ν and resistivity η in a an SB high-resolution numerical simulation. Systematically changing the Reynolds number ($\mathtt{R}$) and the magnetic Prandtl number $\mathtt{mPr} \equiv \nu/\eta$, in some range, these authors aimed at uncovering the trends and scalings of relevant physical properties, notably angular momentum transport, with the relevant non-dimensional number(s).

The most recent calculations in the SB framework with relatively high resolution have been reported by Fromang & Papaloizou (2007) (FP) and by Fromang *et* al. (2007) These studies (both with PBC) focused on the issue of convergence (i.e., numerical resolution) in ideal SB calculations and on ($\mathtt{R}$) and $\mathtt{mPr}$ dependencies of the transport in dissipative conditions.

At the same time King, Pringle & Livio (2007) (KPL), pointed out the discrepancy (of at least one order of magnitude) between the values of α inferred from observations and the estimates of this parameter based on (mainly the early) SB numerical simulations, suggesting that the effective α decreases (almost linearly) with the value of the imposed B_z. They conjectured that perhaps only *zero net flux* (ZF) calculations should be used.

2.1. The problematics of the SB approximation

The essence of the SB approximation is in its *local* approach, that is, the resulting equations for the perturbations on a steady base flow are approximately valid in a small region (a Cartesian box) about a typical point in the disk. The *global* MHD base flow in an almost Keplerian accretion disk is not only rotating and strongly sheared, but it is also inhomogeneous, non-isotropic (endowed with non-zero gradients in density and other physical variables and these gradients have very different scales in different directions) and swirling (streamlines are curved).

Umurhan & Regev (2004) followed a systematic derivation of the approximation using asymptotic scaling arguments with the purpose of quantifying the approximations that are made leading to the SB (see Appendix A of that paper). Two defining nondimensional numbers were stressed in their study- ϵ (the typical disk height h_0, in units of r_0, atypical radius) and δ (the box size in the same units). For details see that paper.

We stress that virtually all numerical calculations that utilize the SB approximation, in all its variants, employ PBC in which the periodicity in the radial direction is sheared (see H95). These sheared-periodic-BC are equivalent to enforcing that all perturbation quantities to be triply (or in the case of a 2-D calculation, doubly) periodic in the *sheared coordinates* system. It is also important that our critique of the SB scheme applies to both FF and ZF conditions.

2.1.1. Relevant length scales

In a typical accretion disk, r_0 is the scale on which curvature terms appear and this is also the scale of any underlying *radial* structure gradients. Vertical stratification obviously appears on the vertical pressure scale-height, which is $\epsilon = h_0/r_0 \ll 1$ in our notation and units. In studies of turbulence it is customary to consider the *injection* scale, $\ell \equiv k_{\rm in}^{-1}$, on which the process of energy injection into the system is effected and the *dissipation* scale, $\equiv k_{\rm d}^{-1}$, on which microscopic viscosity (and in MHD also resistivity) are operative. These are linked by the *inertial* range of scales. Since $k_{\rm in} \ll k_{\rm d}$ (at least in astrophysical systems), the inertial range extends over many orders of magnitude. In MHD turbulence the energy cascades in 2D and 3D are both direct, i.e., from small wave-numbers to large ones, see, e.g., Biskamp (2003) (BI), and consequently the injection scale can be roughly identified with the system's structural scale. In the classical Kolmogorov theory the scale of the largest eddies, which contain most of the energy, is referred to as the *integral* scale. This is also the length scale appearing in the definition of $\mathtt{R}$ and serves as as a measure for the scale over which the turbulent fluctuations are correlated. To avoid complications we shall practically identify the integral scale with the correlation length and the integral scale, denote them as ℓ and refer to them interchangeably. This identification has an obvious intuitive physical basis.

In a disk, which is strongly non-isotropic, two disparate injection scales seem a priori to appear - the horizontal one $\ell_{\rm h} \lesssim r_0$ and the vertical one $\ell_{\rm h} \lesssim \epsilon r_0$ (both given here in dimensional units). It is however reasonable to identify ℓ with the smaller of the two, because this is the size of largest eddies. A quantitative measure of ℓ can be "phenomenologically" determined by the relevant wave-number at which the disturbance kinetic energy $E_{\rm k}(k)$ spectrum peaks. The fact that the scale at which resistivity and viscosity come into play need not be equal makes also the determination of the dissipation scale not unique. Using the Spitzer values we get typically $\mathtt{mPr} \ll 1 \quad (= 6 \times 10^{-29} T^4/\rho \sim 5 \times 10^{-5}$ c.g.s, for CV disk conditions), i.e., $k_\nu > k_\eta$ where k_ν and k_η respectively represent the dissipation scale wave-numbers for viscosity and resistivity. Phenomenologically, the inertial range is considered to be the k region, in which $E_{\rm k}(k)$ exhibits a power-law behavior. Thus, the inertial range in the MHD turbulence of a disk can be considered as including wave-numbers roughly in the range $\epsilon^{-1} \ll k \ll k_\eta$. For a fairly detailed discussion of the MHD turbulence length scales and relevant references see BI. In any case, possible nonlinear interactions between the short vertical wavelength unstable modes with modes (possibly also horizontal ones) whose scale is larger than the SB scale cannot be captured in a SB simulation. It is conceivable that such interactions may play a role in the instability saturation and the development of activity.

2.1.2. Numerical resolution and its relationship to dissipation

Astrophysical fluid systems and their magneto-fluid counterparts (like an accretion disk flow) are endowed with extremely large ($\mathtt{R}$) and magnetic Reynolds ($\mathtt{Rm} = \mathtt{R} \times \mathtt{mPr}$) numbers and therefore numerical simulations cannot yet resolve the full dynamical range of such turbulent flows. In other words, given the present magnitude of computing power, numerical resolution of the full spatial range of these systems, from the system's energy injection scale through the full inertial scale and down to the dissipation scale, is still out of reach.

It has been argued in the past (e.g., BH98) that perhaps a full dynamic range is actually not needed in the accretion disk problem, because the scale of the most unstable *linear* MRI modes is large and in the saturated turbulent state, most of the energy resides in, and the angular momentum transport is done by, the large scale eddies. However, this statement is strongly contested, even for purely hydrodynamical flows. For an extensive, recent, discussion on the use of unresolved hydrodynamical codes for the study of turbulence, see Grinstein, Margolin & Rider (2007). In any case, the conjecture appears to be clearly unfounded for MHD turbulence with very large R. Similar general conclusions also appear in BI. Thus, it appears that a viable estimate of α in a MHD turbulent disk, cannot be obtained from poorly resolved SB simulations. In addition, it is even difficult to quantitatively assess the energetics of MRI driven turbulence from such simulations. The numerical investigations of FP and others actually show, for the FF case that increasing the resolution of simulations relying on numerical dissipation shows a decreasing trend in the calculated transport, suggesting that this is indeed the case.

We may say that systems including turbulent thin accretion disks can be approached numerically in roughly three ways:
(a) Local ideal or/and dissipative calculations.These are a DNS (direct numerical simulation)-like, maximally resolved, calculations, ignoring any scale interaction and gradient or boundary effects.
(b) Global calculations employing some physically motivated sub-grid scale turbulence model. In principle, these LES (large eddy simulations)-like calculations allow confronting accretion disk models with relevant observations (see Bisikalo's contribution in these proceedings).
(c) Global calculations, albeit with unrealistically low Reynolds numbers, - in the purpose of experimenting with the trends of the dynamics with increasing R and/or Rm.

All existing SB simulations nominally belong to the first class, however the great disparity between the smallest resolved scale l_{num} and the true dissipation scales prevent them from being regarded as true DNS. Except for the old α prescription for MRI mediated or induced MHD turbulence, there is as yet no thoroughly tested sub-grid model as might be employed in a LES (or one using some other turbulence model) calculation.

Given the doubt discussed above that numerical dissipation by itself is sufficient in an LES calculation, it seems to be imperative to look for a sub-grid model that is appropriate for the problem at hand. Investigating MRI induced or mediated turbulence in local DNS simulations may certainly offer in-roads into a better understanding of its *local* salient physical features which includes, e.g., some characterization of the emerged turbulence in terms of eddy spectra and correlations. Information of this sort can provide the ingredients for the needed "turbulence model" which would go into a faithful global accretion disk LES (or a simulation employing other turbulent flow computational schemes). Since local simulations are needed to infer the turbulent behavior down to the dissipation scale, rmpC with physically definite boundary conditions seem to be not less appropriate than SB (whose periodic-BC introduce additional difficulties and inconsistencies, see below) for this purpose. Of course, such simulations must be fully resolved (DNS) and pushed to the maximum Re and Rm values in order for them to yield this kind of information.

2.1.3. Symmetry and boundary conditions

The SB approximation with PBC is endowed with a particular symmetry: it allows for shearwise (in the Cartesian geometry of the box it corresponds to the x-direction) invariant solutions.

The x-invariance symmetry, which allows for the existence of special solutions (channel modes) in the SB, is obviously broken by any non-periodic-BC on x, even in local SB analyses and this is effected on the box scale. The

SB scale, appropriate for thin accretion disks satisfies $\epsilon^2 \cdot r_0 < L < \epsilon \cdot r_0$. Radial symmetry breaking by the above mentioned global physical effects are manifested on a length scale of the order r_0, but non-PBC are operative on the smaller box scale.

We finally turn to the most important observation about the use of PBC in numerical studies of turbulence and its bearing on SB simulations. Clearly, space PBC are not accessible in realistic physical situations, and they only can be faithfully employed in the study of homogeneous turbulence when one assumes that real boundary effects are not important (Dubois *et* al.1999). The caveats pertaining to the use of periodic-BC in numerical simulations of turbulence are discussed in detail in §7.2 of Davidson (2004) from which we only bring here issues that are explicitly relevant to SB simulations of MHD turbulence in an accretion disk. In order for the bulk of the turbulence not to be seriously affected by the imposed periodicity, the box size should satisfy $L \gg \ell$, otherwise the "tails" of the relevant correlation functions (of the turbulent fluctuations) are not only cut off, but also very significantly lifted (see figure 7.6 of Davidson, 2004). This crucially affects the behavior of the *large* scale dynamics. If we accept the reasonable conjecture that the injection scale relevant for an accretion disk is indeed $\ell = \ell_{\rm v} \lesssim h_0 = \epsilon r_0$, the *minimal* size of the SB must be the disk height. But then z periodic-BC cannot be used and we are only left with the option of semi-global or global simulations. In the next Subsection we shall demonstrate, with the help of a numerical experiments, that using a SB, whose size is not large enough (as compared to the scale of the relevant dynamical structures), and PBC, gives rise to spurious energy fluctuations, casting a serious doubt on the physical significance of the simulation results.

2.1.4. Energetics of the SB

In any continuum mechanics problem posed in a finite domain, as is the case for SB MHD numerical calculations of the kind discussed here, it is generally advantageous to consider integral physical conservation statements, valid on the domain. A particularly useful consideration of this kind, in this problem, arises from the examination of the energy budget for the system of equations

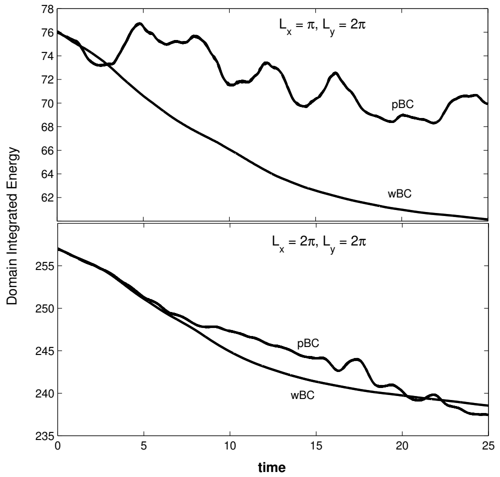

Fig. 1. The total energy (in arbitrary units) in the SB as a function of time (in units of Ω_0^{-1}). In the upper panel the size of the SB is $L_x = \pi$ and $L_y = 2\pi$ and the result of the calculation with PBC is compared with the one using WBC. In the lower panel the same is shown for a SB having a double shearwise extent. In both cases the simulations are started with the same initial conditions

We have performed a number of numerical experiments in the purpose of examining the effects of the kind of BC employed (periodic versus wall) and of the SB size (relatively to a relevant integral scale ℓ of the problem) on the evolution of the SB energy content. We have chosen to focus on the total integrated energy of the domain, as this quantity has (at least to us) the most definite and best understood physical meaning.

We shall consider the evolution of the total domain energy for two sets of boundary conditions. The first of these are the PBC, as laid out before in the paper. The second type of BC are appropriate for a rotating channel and we refer to them as WBC They implement at the shearwise boundaries i) no normal flow ii) no normal magnetic field flux and iii) free-slip boundary conditions. i.e.

$$u = 0,\ b_x = 0,\ \partial_x v_y = 0, \text{ at } x = \pm L_x/2. \quad (1)$$

WBC are periodic in the azimuthal direction. Note that the *equations* solved are identical for both sets of BC.

The description of the results can be seen in the figure caption. More details and the whole contentent of Section 2 (which is rather schematic due to the lack of space) appear in Regev & Umurhan (2008).

3. Saturation of the MRI

We have also performed a weakly nonlinear analysis of the MRI near threshold for a rotating flow in channel geometry, subject to idealized but mathematically expedient boundary conditions. This kind of approach is important because the viability of this linear instability as the driver of turbulence and angular-momentum transport relies on understanding its nonlinear development and saturation. By complementing the above mentioned simulations, analytical methods remain useful to gain further physical insight. To facilitate an analytical approach we make a number of simplifying assumptions so as to make the system amenable to well-known methods (e.g. Cross & Hohenberg, 1993) for the derivation of nonlinear envelope equations (i.e. with an amplitude weakly dependent on time *and* space). Additionally we assume a narrow channel geometry, i.e. the small-gap limit of Taylor-Couette flow.

The hydromagnetic equations in cylindrical rotating coordinates are applied to the neighborhood of a representative point in the system, the base flow is steady and incompressible with constant pressure. The velocity $\mathbf{V} = U(x)\hat{\mathbf{y}}$ has a linear shear profile $U(x) = -qU_0 x$, representing an axisymmetric flow about a point r_0, that rotates with a rate Ω_0 defined from the differential rotation law $\Omega(r) \propto \Omega_0(r/r_0)^{-q}$. We restrict ourselves to a constant initial magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. Cartesian coordinates x, y, z are used to represent the radial (shear-wise), azimuthal (stream-wise) and vertical directions, respectively. The base flow is axisymmetricaly disturbed by perturbations of the velocity, $\mathbf{u} = (u_x, u_y, u_z)$, magnetic field, $\mathbf{b} = (b_x, b_y, b_z)$, and total pressure, ϖ.

The end result of the rather lengthy asymptotic procedure procedure. see Umurhan *et* al. 2007b (URM), is the well-known real Ginzburg-Landau Equation (rGLE) which, for $\mathtt{rPm} \ll 1$, is

$$\partial_T A = \lambda A - \frac{1}{\mathtt{rPm}C} A|A|^2 + D\partial_Z^2 A. \qquad (2)$$

Here A ia proportional to the amplitude of the perturbation, while the other symbols are constants related to the physical parameter values, see Umurhan *et* al. 2007a (UMR).

The key results of this idealized analysis are the saturation valuse,

$$A \to A_s \sim \sqrt{\mathtt{rPm}\,C}. \qquad (3)$$

This gives a fixed energy for the region in question and the following scaling of the AMT

$$\dot{\mathtt{J}} \sim \mathtt{R}^{-1}, \qquad (4)$$

for $\mathtt{rPm} \ll 1$.

We have also performed numerical calculations, using a 2-D spectral code to solve the original nonlinear equations, in the stream-function and magnetic flux function formulation, near MRI threshold. The asymptotic theory reflects the trends seen in these simulations.

The numerical and asymptotic solutions developed here show that in the saturated state the second order azimuthal velocity perturbation becomes dominant over all other quantities for $\mathtt{rPm} \ll 1$ and appears to be the primary agent in the nonlinear saturation of the MRI in the channel. It acts anisotropically so as to modify the shear profile and results in a non-diagonal stress component (relevant for radial angular momentum transport), which behaves like $\sim 1/\mathtt{R}$. For a detailed exposition of this asymptotic work which shows that the MRI saturates asymtotically for low $\mathtt{mPR}$ numbers see UMR and URM.

This analysis is complementary to that of Knobloch and Julien (2005) who have reported the results of an asymptotic MRI analysis for a developed state far from marginality.

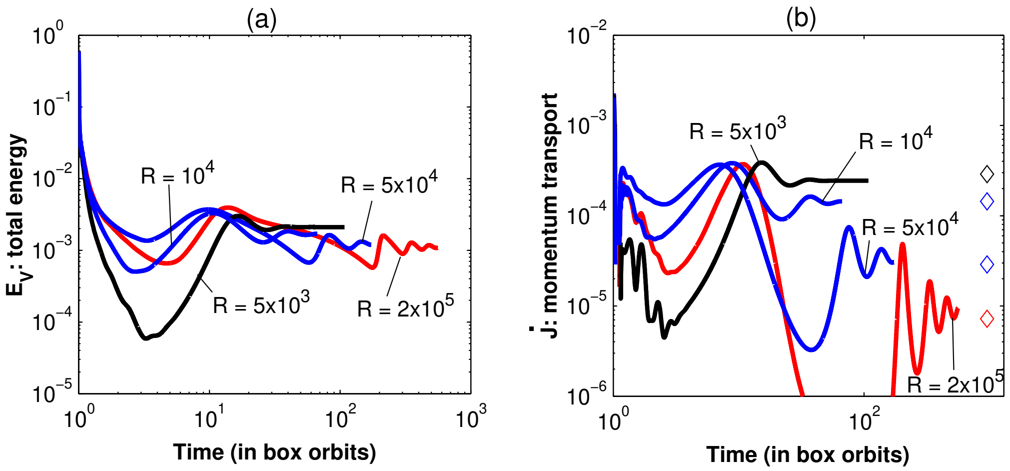

Fig. 2. E_V [panel(a)] and $\dot{J}$ [panel(b)] as a function of time from numerical calculation (solid lines) for Reynolds numbers $\mathrm{R} = 5 \times 10^3, 10^4, 5 \times 10^4, 2 \times 10^5$. The diamonds in panel (b) show the scaling predicted by our asymptotic analysis, which also predicts a constant final value of the disturbance energy, as apparent in panel (a). Given that $\dot{J} > 0$, it means that the angular momentum transport is outwards (positive x direction). One "box" orbit $= 2\pi$ in non-dimensional units

4. Summary

Based on the arguments of the work described here and in Section 2 we have no choice but to conclude that decades of numerical simulations of the "nonlinear development" resulting from the MRI in disks has not brought us new and significant knowledge on AMT in accretion disks as are though to exist in CVs.

Also laboratory experiments have so far not yielded any conclusive indications of MRI driven turbulence, save for the case in which relatively strong azimuthal fields are applied In thin accretion disks, however, strong toroidal (or any other, for that matter) magnetic fields cause significant magnetic pressure and are likely to be inappropriate for accretion disks in the thin, rotationally supported, disk limit.

It is not easy to answer the question what kind of numerical scheme or which setup is best suited for the problem. In any case, it is not clear what is less appropriate for local numerical studies of accretion disks: WBC or PBC in the SB. We believe that the use of PBC, especially in the radial direction, suffers from serious problems unless the power in the perturbation fields is localized away from the boundaries of the system (Dubois *et* al. 1999; Davidson, 2004).

It seems to us that the logical approach to the accretion disk angular momentum transport problem should thus consist of

1. Mathematically and physically sound calculations, most likely in the mrpC (magnetic rotating plane Couette) or mTC (magnetic Taylor Couette) thin-gap setups, in the purpose of learning about the detailed local properties of MHD turbulence (its energy spectrum, correlations etc.)
2. Devising a turbulence model, based on the above, so as to replace the α model.
3. Global calculations employing the above turbulence model(s) and various boundary conditions, giving rise to accretion disk models which can be confronted with observations.

The problem of angular momentum transport in accretion disks, especially if MHD turbulence in a non-isotropic strongly sheared medium is the physical mechanism responsible for it, is extremely difficult and involved. This work, as well as that of KPL and others, indicates that the understanding gained so far from

SB and other numerical simulations seems unfortunately not to be adequate enough to provide a viable physical prescription that can be effectively used in modeling accretion disks, beyond the old α-viscosity parametrization.

We think that only an extensive and coherent program, taking also into account recent developments and ideas of transient growth, may shed a new light on the difficult problem of turbulence in AD. The lesson to be learned from that failure of the MRI driven turbulence idea, is, in our opinion, that the study has to be conducted in a collaborative and completely sincere and open manner - incorporating and taking account the efforts of different groups and approaches.

5. Discussion

DIMITRI BISIKALO: In purely hydrodynamic models there were suggestions of transient growth. But it is bounded.
ODED REGEV: Not much specific is known. It is hoped that some finite perturbations may grow to sizes, in which nonlinear interaction between them may play a role. There is no definite model. Just an idea

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