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Astrometric reference frames in the solar system and beyond

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Abstract. This presentation discusses the mathematical principles of constructing coordinates on curved spacetime manifold in order to build a hierarchy of astrometric frames in the solar system and beyond which can be used in future practical applications as well as for testing the fundamentals of the gravitational physics - general theory of relativity.

1. Introduction

Fundamental astrometry is an essential ingredient of modern gravitational physics. It measures positions and proper motions of various celestial objects and establishes a correspondence between theory and observations. Theoretical foundation of astrometry is general theory of relativity which operates on curved spacetime manifold covered by a set of local coordinates which are used to identify positions and to parametrize the motion of celestial objects and observer. The set of the local coordinates is structured in accordance with the hierarchical clustering of the gravitating bodies. The astrometric frame is derived after making observational connection between the theoretical and observed coordinates of the celestial objects. The frame can be either dynamical or kinematic depending on the class of celestial objects having been used for building the frame. The local astrometric frames are not fully independent – they must be matched to a global astrometric frame which is usually build by making use of quasars as benchmarks. Mathematical description of the local and global frames require to solve Einstein's equations in order to find out the metric tensor corresponding to the coordinate charts covering either a local domain or the entire spacetime manifold.

The standard theory adopted by the International Astronomical Union (IAU) postulates that spacetime is asymptotically-flat and the only source of gravitational field is the matter of the solar system (Soffel et al. 2003; Kopeikin 2007). This approach completely ignores the presence of the huge distribution of mass in our own galaxy – the Milky Way, the local cluster of galaxies, and the other visible matter of the entire universe. Moreover, the current mathematical approach to the construction of astrometric local and global coordinates does not take into account the presence of the dark matter and the dark energy which, altogether, make up to 96% of the total energy of the observed universe. It requires a certain extension of the current IAU paradigm on astrometric frames to account for recent astrophysical discoveries. The updated astrometric-frame theory has to answer another important question about how the Hubble expansion of the universe affects our measurement of the spacetime properties. This problem has been extensively discussed recently by Krasinsky &

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Brumberg (2004), and especially by Carrera & Giulini (2010) on the basis of exact solutions of Einstein's equations. However, their arguments can not be fully accepted so far as the post-Newtonian approximation methods have to confirm (or disprove) their mathematical derivations.

2. Astrometric frames in the solar system

The current IAU paradigm suggests that the solar system is the only source of gravitational field and that at infinity the spacetime is asymptotically flat. This paradigm has been worked out over time to the degree of practical implementation by a number of researchers. The most comprehensive theoretical description of the IAU reference frames paradigm is presented in (Soffel et al. 2003) and in a recent monograph (Kopeikin et al. 2011). The IAU paradigm operates with a number of local and one global reference frames which are connected to each other by post-Newtonian coordinate transformations. The most important for description of astrometric measurements is the local frame of observer, the geocentric frame (GRF), and the barycentric frame (BRF) of the entire solar system (Kopeikin 2011).

2.1. The proper frame of observer

An ideal observer in general relativity must be understood as a point-like test particle equipped with measuring devices like clocks, rulers, gyroscopes, lasers, receivers, and other possible measuring devises (Klioner 2004). Observer moves along timelike worldline which may be or may be not a geodesic. For example, observer located on the surface of the Earth is subject to Earth's gravity force that incurs non-geodesic acceleration of gravity $g = 9.81$ m/s^2. On the other hand, observer located on board of a drag-free satellite is not subject to any external force, and moves along a geodesic. The observer is always located at the origin of the local (topocentric) frame. He measures the proper time τ with the help of the ideal clock, and the proper length ℓ with the help of the ideal rules calibrated by means of light (laser) pulses which he sends to and receives back from retroreflectors and or transponders placed on other celestial bodies. Spatial axes of the observer's local frame are not rotating in dynamic sense and their orientation is maintained by the gyroscopes. The proper time, τ, and the spatial axes, ξ^i, of the local frame are considered as four-dimensional coordinates $\xi^\alpha = \{\tau, x^i\}$.

Relative motion of observer is usually measured, and is supposed to be well-known, with respect to another local frame associated with the massive body at the surface of which (or by which) the observer is located. The most theoretically-elaborated example of such a local frame is the geocentric frame.

2.2. The geocentric reference frame

Geocentric reference frame is a local frame parametrised with coordinates $w^\alpha = \{u, w^i\}$ where u is the coordinate time and w^i are spatial coordinates with origin at the center of mass of the earth. The spatial axes have no rotation in kinematic sense with respect to the BRF, and stretch over space out to the distance being determined by the acceleration of the world line of geocenter (Kopeikin et al. 2011).

Besides the geocentric frame, there exists a number of other local frames in the solar system associated with the massive planets and their satellites (moons). These frames are required to describe the orbital motion of planetary satellites, for spacecraft's navigation, or for measuring gravitational field of the planets. The geocentric and all planetocentric references frames are intimately related to a single coordinate chart known as the barycentric reference frame of the solar system.

2.3. The barycentric reference frame

The barycentric reference frame of the solar system is parametrized with coordinates $x^\alpha = \{t, x^i\}$ having the origin at the center of mass of the solar system. Because the spacetime is assumed to be asymptotically-flat the spatial coordinate axes of the barycentric frame go to (spatial) infinity. Moreover, they have no kinematic rotation or deformation. Practical realization of this frame is given in the form of ICRF that represents a catalogue of several hundred radio quasars uniformly distributed over the whole sky (Lanyi et al. 2010).

2.4. Transformation between the frames

Coordinates corresponding to the astrometric frames of the solar system are connected to each other by a smooth post-Newtonian transformations defined in the domain of overlapping the coordinate charts. These transformations generalize the Lorentz transformation of special theory of relativity by taking into account the effects of the gravitational field of the solar system objects.

The IAU paradigm of the astrometric reference frames works very well in the solar system. However, it remains unclear to what extent the paradigm can be hold beyond the solar system. The problem is that the solar system is not fully isolated from the external environment – it moves around the center of our own galaxy with some finite acceleration. Moreover, a more important argument is that the spacetime is not asymptotically flat but is described by the Robertosn-Walker metric of the expanding universe. It motivates us to extend the framework of the IAU paradigm and to build the coordinates associated with the galaxy and with the expanding universe.

3. The galactocentric reference frame

Definition of the galactocentric coordinates $X^\alpha = \{T, X^i\}$ is a fairly straightforward generalization of the IAU paradigm. We consider now the whole galaxy as an isolated system with asymptotically flat spacetime. Then, the barycentric reference frame of the solar system becomes one of the local frames associated with stars or clusters of stars in the galaxy. We put the origin of the galactocentric coordinates at the center of mass of the Milky Way and postulate that the coordinate spatial axes do not rotate in kinematic sense with respect to quasars. Transformation between the barycentric coordinates of the solar system and the galactocentric coordinates is given by

$$t = T - c^{-2}\left[B(T) - V^i_{\scriptscriptstyle B}\left(X^i - X^i_{\scriptscriptstyle B}\right)\right] , \qquad (1)$$

$$x^i = L^i{}_j\left(X^j - X^j_{\scriptscriptstyle B}\right) , \qquad (2)$$

where $V^i_{\scriptscriptstyle B}$ is the velocity of the barycenter with respect to the center of mass of the Milky Way ($V_{\scriptscriptstyle B} \simeq 250$ km/s), and the matrix

$$L^i{}_j = \left(1 + \frac{U_{gal}}{c^2}\right)\delta^{ij} + c^{-2}\left(\frac{1}{2}V^i_{\scriptscriptstyle B}V^j_{\scriptscriptstyle B} + F^{[ij]}\right) , \qquad (3)$$

describes the Lorentz and gravitational length contractions due to the velocity V^i and the gravitational potential U_{gal} of the Milky Way as well as a rotation of spatial axes of the barycentric frame with respect to the axes of the galactocentric frame (the term with $F^{[ij]}$). The orbital motion of the solar system with respect to the galactocentric coordinates is slow. Hence, all quantities depending on the position of the barycenter of the solar system, $X^i_{\scriptscriptstyle B}$ can be expanded in the polynomial series of time. For example [1],

$$X^i_{\scriptscriptstyle B}(T) = X^i_{\scriptscriptstyle B}(T_0) + V^i_{\scriptscriptstyle B}T + \frac{1}{2}A^i_{\scriptscriptstyle B}T^2 + \ldots , \qquad (4)$$

$$B(T) = B(T_0) + \dot{B}(T_0)T + \frac{1}{2}\ddot{B}(T_0)T^2 + \ldots , \qquad (5)$$

and so on. Here,

$$\dot{B} = \frac{1}{2}V^2_{\scriptscriptstyle B} + U_{gal} , \qquad (6)$$

$$\ddot{B} = \boldsymbol{V}_{\scriptscriptstyle B} \cdot \boldsymbol{A}_{\scriptscriptstyle B} + \dot{U}_{gal} . \qquad (7)$$

The velocity-dependent term in (1) causes the change in the rate of the barycentric time (TCB) as measured at distance $R = |X^i - X^i_{\scriptscriptstyle B}|$ from the barycenter. For terrestrial observer it reveals as an annual periodic variation amounting to 0.37 s. Nevertheless, this large relativistic effect is extremely difficult to observe as it is hidden in the astrometric positions of stars in the form of the secular aberration (Kopeikin & Makarov 2006). Function $B(T)$ in (1) causes divergence of TCB from the more uniform, galactocentric time T. The derivative $\dot{B} \simeq 8.4 \times 10^{-7}$ redefines the unit of time (SI second) as measured in TCB. It causes re-scaling the astronomical unit of length and mass. This point has been never discussed by the IAU, and its practical consequences for astrometric observations are not quite clear. The second derivative $\ddot{B} \simeq 2 \times 10^{-15}$ yr^{-1} causes systematic

[1] An overdot denotes a time derivative. $A^i_{\scriptscriptstyle B} = \dot{V}^i_{\scriptscriptstyle B}$

quadratic drift of TCB with respect to the uniform galactic time. It may be important in precise astrometric observations of quasars to ensure the stability of the ICRF and the space astrometry catalogues. The rate of relativistic precession of the spatial axes of the barycentric frame of the solar system with respect to the galactocentic frame, is small and amounts to 0.004 μas/yr^{-1}. It can be ignored in processing of astrometric observations.

4. Astrometric frame in cosmology

Cosmological observations have established that the universe is expanding with acceleration. Current cosmological paradigm explains this acceleration by the presence of the dark energy. The nature of the dark energy is yet unknown but the common wisdom is that it is made of a scalar field with self-interaction described by some form of potential. Cosmological observations also reveal the presence of the dark matter in halos of galaxies and the clusters of galaxies. The overall spacetime in cosmology is curved and time-dependent with space cross-sections being flat. The background Friedmann-Lemetre-Robertson-Walker (FLRW) metric $\bar{g}_{\alpha\beta}$ of the universe is conformal to the flat spacetime. In the isotropic coordinates the FLRW metric reads

$$\bar{g}_{\alpha\beta}(\eta) = a^2(\eta)\eta_{\alpha\beta} \ , \tag{8}$$

where $a(\eta)$ is the scale factor depending on the conformal time η, and $\eta_{\alpha\beta}$ is the Minkowski metric. The presence of the solar system (and other massive bodies - stars, galaxies, etc.) causes perturbation of the background metric

$$g_{\alpha\beta}(\eta, \boldsymbol{x}) = a^2(\eta)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) \ , \tag{9}$$

which should be compared with the current post-Newtonian paradigm of the IAU that postulates that the scale factor $a(\eta) \equiv 1$. This postulate is not valid, at least, in principle, thus, challenging theorists for deeper exploration of the impact of cosmological expansion on astrometric measurements and on dynamics of self-gravitating systems like the solar system, binary pulsars, galaxies and their clusters.

The task is to calculate the perturbation $h_{\alpha\beta}$ by solving Einstein's equations. Physical cosmology has developed a powerful approximation scheme for doing this. Unfortunately, it can not be applied for calculation cosmological perturbations caused by self-gravitating isolated systems. This is because physical cosmology assumes the density contrast of the perturbation, $\delta\rho/\bar{\rho} \ll 1$, where $\bar{\rho}$ is the mean density of the universe [2]. However, the density contrast in isolated astronomical system, $\delta\rho/\bar{\rho} \gg 1$, that makes unacceptable any approximation method previously developed in cosmology to find out the metric perturbation $h_{\alpha\beta}$ for the purposes of celestial mechanics and astrometry.

To overcome the problem some researchers resorted to exact methods of embedding a spherically-symmetric Schwarzschild solution to the expanding FLRW universe. The most famous model was worked out by Einstein & Straus (1945, 1946) and is known as the *Swiss cheese* or *vacuole* solution (Schücking 1954) in a dust-dominated universe. The vacuole solution is unsatisfactory from theoretical point of view since it is unstable and can be destroyed by imposing an infinitesimal perturbation. Furthermore, the vacuole model does not provide a realistic model of spacetime at the scales below the scale of galaxy clusters. This is because the radius of the vacuole for a central object like a star or a galaxy significantly exceeds the average distance between stars/galaxies (Bonnor 2000).

For this reason other exact solutions were sought. However, the complexity and nonlinearity of Einstein's equations make it a difficult task to construct a suitable exact solution which may serve as realistic model for actual physical situations. Indeed, as a rule, exact solutions of Einstein's equations are known for idealized situations typically admitting high degree of symmetry. At the same time, realistic environment of isolated gravitational systems has no symmetry. The problem then is how to immerse the non-symmetric (and time-dependent) localized distribution of mass smoothly to the expanding FLRW universe. The predominant idea was to glue several exact solutions across suitably chosen hypersurface with the appropriate matching conditions imposed on the metric tensor and its derivatives.

[2] $\bar{\rho} \simeq 10^{-30}$ g/cm^3 in the present epoch

Such an approach was pioneered by McVittie (1933) who has taken the Schwarzschild solution and embedded it to a spatially-flat FLRW universe by multiplying the spatial part of Schwarzschild's metric by a time-dependent scale factor $a(t)$. Such "marriage" of two metrics is possible, if and only if, the mass of the central body is a function of time, $m = m_0/a(t)$, where m_0 is a constant mass. Scrutiny analysis of Einstein's equations reveals that in the most general case, McVittie's metric is physically implausible because imposing the condition of a non-vanishing central mass m_0 and the time-dependence scale factor $a(t)$ of the FLRW spacetime leads to an unrealistic equation of state of matter in universe. Similar to the case of the vacuole solution, the McVittie metric is spherically-symmetric and, hence, inappropriate for analysing more realistic situations having no spherical symmetry like binary pulsars or planetary systems.

The only way to surpass mathematical difficulties of embedding an isolated astronomical system to the cosmological environment is to resort to a suitable method of approximations which generalizes the post-Newtonian approximations in asymptotically-flat spacetime to the case of curved and expanding cosmological manifold.

5. Post-Newtonian approximations in cosmology

We have worked out a new method of finding cosmological perturbations induced by an isolated astronomical system (Kopeikin & Petrov 2012) which is valid for sufficiently high contrasts of matter density but singularities, and for arbitrary equation of state of the background matter. It is based on the Lagrangian theory of perturbations of an arbitrary-curved and dynamically-evolving spacetime manifold (Grishchuk et al. 1984; Popova & Petrov 1988). In our approach both the universe and the localized source of perturbation are mathematically modelled by a Lagrangian depending on a set of dynamic variables which include the metric perturbation, $h_{\alpha\beta}$, and several scalar fields $\Phi, \Psi, \ldots$. The metric perturbation describes the gravitational field of the isolated astronomical system along with the perturbation of the background universe. The perturbation of the background geometry couples with the source of gravitational field of the isolated system and makes the effective mass of the system depending on time even if the bare mass of the system was taken as constant. This resembles and may explain the physical origin of the time-dependence of the central mass in the McVittie metric. The scalar fields describe the structure of matter. In particular, the dark matter is modelled as a perfect fluid described by the Clebsch potential, Φ, that is the specific enthalpy, μ, of the fluid (Schutz 1970). The dark energy is modelled as a quintessence scalar field, Ψ, with an arbitrary potential $W(\Psi)$. Other scalar fields being pertinent to the problem may be accounted for, if necessary. The localized isolated system is also described by a Lagrangian of a continuous distribution of matter. It serves as a source of gravitational field of the isolated system that is a bare perturbation of the background cosmological spacetime.

The overall Lagrangian of the system is presented as a Taylor (asymptotic) series with respect to the perturbations of all variables,

$$\mathcal{L} = \bar{\mathcal{L}} + \mathcal{L}_1 + \mathcal{L}_2 + \ldots , \tag{10}$$

where $\bar{\mathcal{L}}$ is the Lagrangian of the background FLRW universe, and $\mathcal{L}_i$ ($i = 1, 2, \ldots$) are linear, quadratic, etc. perturbations. The field equations for the metric perturbation and the dynamic matter variables are obtained by taking variational derivatives from $\mathcal{L}$ with respect to the variables $h_{\alpha\beta}$, Φ, Ψ, etc. We do not impose any limitation on the equation of state of the background FLRW universe so that the theory is applicable for wide range of cosmological models.

As an example, we write down the system of the field equations defining the metric tensor perturbations in case of the universe filled up with the dark matter in the form of dust and the dark energy in the form of the cosmological constant Λ. They are (Kopeikin & Petrov 2012)

$$\begin{aligned} \Box q - 2H\bar{v}^{\alpha} q_{,\alpha} + \left(H^2 - a^2\Lambda\right) q &= \\ 8\pi a^2 \left[\sigma + \bar{\rho}_{\rm m}\left(\delta_{\rm m} + a^{-1}H\chi_{\rm m}\right)\right] &, \end{aligned} \tag{11}$$

$$\Box p_{\alpha} - 2H\bar{v}^{\beta} p_{\alpha,\beta} + \left(H^2 - a^2\Lambda\right) p_{\alpha} = 16\pi a \sigma_{\alpha} ,$$

$$\Box p_{\alpha\beta} - 2H\bar{v}^{\gamma} p_{\alpha\beta,\gamma} = 16\pi\sigma_{\alpha\beta} ,$$

where $H = \dot{a}/a$ is the Hubble parameter, $\Box = \eta^{\mu\nu}\partial_{\mu\nu}$ is the wave operator in flat spacetime, comma denotes a partial derivative with respect to a corresponding coordinate, $q = \bar{v}^{\alpha}\bar{v}^{\beta} l_{\alpha\beta}$, $p_{\alpha} = -\bar{\pi}_{\alpha}{}^{\beta}\bar{v}^{\gamma} l_{\beta\gamma}$, $p_{\alpha\beta} = \bar{\pi}_{\alpha}{}^{\mu}\bar{\pi}_{\beta}{}^{\nu} l_{\mu\nu}$, $l_{\alpha\beta} = -h_{\alpha\beta} + (h^{\mu}{}_{\mu}/2)\eta_{\alpha\beta}$, and $\bar{\pi}_{\alpha}{}^{\beta} = \delta^{\beta}_{\alpha} + \bar{v}_{\alpha}\bar{v}^{\beta}$ is the operator of projection on the hypersurface being orthogonal to 4-velocity v^{α} of observer with respect to the FLRW isotropic frame. The source of the gravitational perturbations is the tensor of energy-momentum, $\tau_{\alpha\beta}$, of the localized system, and

$$\begin{aligned} \sigma &= \bar{v}^{\alpha}\bar{v}^{\beta}\tau_{\alpha\beta} + \bar{\pi}^{\alpha\beta}\tau_{\alpha\beta} , \qquad (12) \\ \sigma_{\alpha} &= -\bar{\pi}_{\alpha}{}^{\beta}\bar{v}^{\gamma}\tau_{\beta\gamma} , \\ \sigma_{\alpha\beta} &= \bar{\pi}_{\alpha}{}^{\mu}\bar{\pi}_{\beta}{}^{\nu}\tau_{\mu\nu} . \end{aligned}$$

We notice that the source of the scalar gravitational perturbation q includes two more functions, $\delta_{\rm m}$ and $\chi_{\rm m}$, which describe the density perturbation of the background FLRW spacetime. Equations for these functions are as follows (Kopeikin & Petrov 2012)

$$\bar{v}^{\alpha}\bar{v}^{\beta}\delta_{{\rm m},\alpha,\beta} + H\bar{v}^{\alpha}\delta_{{\rm m},\alpha} - 4\pi a^2\bar{\rho}_{\rm m}\delta_{\rm m} = 4\pi a^2\sigma \quad , \qquad (13)$$

$$\Box\chi_{\rm m} + \frac{1}{2}\left(3H^2 - a^2\Lambda\right)\chi_{\rm m} = a\bar{v}^{\alpha}\delta_{{\rm m},\alpha} \quad , \qquad (14)$$

where $\bar{\rho}_{\rm m}$ is the mean density of the background universe. The perturbation $\delta_{\rm m}$ is induced by the presence of the localized system. However, it can be generated by the primordial cosmological perutbations as well. The impact of the Hubble-induced density $\bar{\rho}_{\rm m}$ on the measured mass of the isolated system is small. However, it is accumulated over time starting from the origination of the progenitor of the isolated system making significant contribution to the current value of the observed mass.

6. Connection to astrometric observables

The connection of the theory to astrometric observations is not straightforward. First of all, we notice that q is to be an analogue of the Newtonian potential but solution of (11) does not yield the Newtonian potential right away. This is because of the presence of the scale factor a in the right side of equation (11) which makes the solution dependent on cosmological time. The magnitude of the "Newtonian" potential q evolves in the conformal coordinates as the universe expands – the effect which can be interpreted as time-dependence of the universal gravitational constant G in the spirit of the Dirac large numbers hypothesis. However, the conformal coordinates are not observable quantities nor any other coordinates. The relationship between the coordinates and observations is established by mean of propagating photons between worldline of observer and a spacetime event. The round trip of photon is described by solution of light geodesics which is given in the first approximation and in the conformal coordinates by the same equation as in flat spacetime,

$$x^i = k^i(\eta - \eta_0) + x^i_0 . \qquad (15)$$

Here, k^i is the unit vector in the direction of propagation of photon ($\delta_{ij}k^ik^j = 1$), η_0 is the (conformal) time of emission of photon from the point with coordinates x^i_0. The conformal time, η, relates to the proper time of observer, τ, by an ordinary differential equation,

$$\frac{d\tau}{d\eta} = a(\eta)\left(1 - \beta^2 - h_{\mu\nu}\beta^{\mu}\beta^{\nu}\right)^{1/2} , \qquad (16)$$

where $\beta^i = dx^i/d\eta$ is the velocity of observer expressed in the conformal coordinates. Solution of this equation depends on the scale factor $a(\eta)$. Hence, the radar distance $\ell = c(\tau_3 - \tau_1)$ between observer and the point in space is a rather complicated function of the conformal coordinates and the scale factor $a(\eta)$. The overall technical details of the calculation of observables along with the discussion of the post-Newtonian dynamics in the expanding universe are given in (Kopeikin & Petrov 2012).

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