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Special frames in General Relativity: applications to the 1PN approximation

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Abstract. After reviewing the basic steps underlying any measurement process in general relativity we discuss certain freedom allowed by the theory, namely the choice of observers and that of special frames adapted to the observer world lines. The arbitrariness in the choice of spatial frames is often removed by selecting those frames which may have an operational definition. In this context, a special role is played by Fermi-Walker transported frames which have their physical realization in a set of three mutually orthogonal gyroscopes. Along a geodesic orbit the Fermi-Walker transport law reduces to parallel transport. We give here the necessary prescriptions to explicitly construct a triad of gyro-fixed axes along an accelerated world of a 1PN approximated metric.

1. Introduction

In general relativity any measurement process should be considered carefully, for a number of reasons. The increased dimensions and the nonvanishing curvature of the spacetime, in fact, complicate matters and drastically change the situation in comparison with the Newtonian case as well as the special relativistic one. However, if the physical laws have their natural 4-dimensional formulation, i.e. they are "at home" in the spacetime, any specific measurement —performed by an arbitrary observer— deals with events happening in a certain space point and having a certain duration in time. That is, the unifying aspect of the spacetime is lost when operating with an observer, whatever he/she is and whatever his/her kinematical conditions are.

Actually, one may establish few basic steps underlying the measurement process (de Felice & Bini D 2010):

1. Specify the phenomenon under investigation.
2. Model that phenomenon, i.e., identify the covariant equations which give a fully satisfactory description of it.
3. Select the observer (or the observer family) who makes the measurements.
4. Chose an observer-adapted-frame, i.e., start the spacetime splitting into the observer's space and time.
5. Understand the locality properties of the measurement under consideration, i.e., understand if the measurement is local or

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non-local with respect to the background curvature.

6. Identify the frame components of those quantities which are the observational targets.
7. Find a physical interpretation of the above components following a suitable criterion such as, for example, comparing with what is known in special relativity or in non-relativistic theories.
8. Verify the degree of the residual ambiguity in the interpretation of the measurements and decide, eventually, the strategy to eliminate it.

Evidently each step of the above procedure relies on the previous one, and hardly one may think to eliminate some of them. Furthermore, let us recall that fixing the observer is independent of the coordinate choice which has been used to represent the spacetime metric: it is possible either to have many observers within a given choice of the coordinate grid or deal with the same (given) observer within many coordinate systems. Whatever is the choice of the coordinate system, one may adapt to a given observer many spatial frames, each providing a different perspective. A measurement requires a mathematical modeling of the target but also of the measuring conditions which account for the kinematical state of the observer and the level of accuracy of his measuring devises. The physical interpretation of a measurement, in turn, requires some previous knowledge about the same object of investigation (see (de Felice & Bini D 2010) for a detailed discussion).

To make a long story short, when dealing with any measurement problem in general relativity one needs first to clarify what is going to be measured (i.e., the observable) and then who is going to perform the requested measurement (i.e., the observers). General relativity leaves as free the choice of the observers, differently from special relativity where inertial observers play a special role. Furthermore, once observables and observers have been indicated, the assessment of the measurement apparatus can still be defined with the consequent choice of the most suited reference frame, namely the reference axes adapted to the observer world line.

If the gravitational context of the problem is already specified, a "suggestion" for the frame choice may come from the background geometry itself. Otherwise, along the observer world lines there exist certain intrinsic choices of frames. In the former case, for instance one can align axes according with the spacetime symmetries, like Killing directions; in the latter case, instead, one may take advantage of either Frenet-Serret formalism (or variation of this; see Bini, de Felice, & Jantzen (1999) for the definition of relative Frenet-Serret frames) or geometric transport laws along the observer world lines, like Fermi-Walker or Lie transport. Fermi-Walker dragged axes are of special importance because they have their physical realization in a set of mutually orthogonal gyroscopes, easy to construct also from a technical and mechanical point of view. When the considered orbit reduces to a geodesic the Fermi-Walker transport law coincides with parallel (or Levi-Civita) transport law.

In general, if the spacetime is arbitrary enough (i.e., without special symmetries) and the world line along which either the Fermi-Walker or parallel transport law is considered is an arbitrary world line (or geodesic), the problem of explicitly determining such an adapted frame is hopeless. Generic spacetime metric at 1PN order, which we are going to discuss in the present paper, allow instead for closed form expressions. To the best of our knowledge, a general discussion in this sense is original and may have also a number of applications, either in a pure theoretical context or in the framework of high precision astrometry, GPSs and navigation in space.

2. Spacetime metric at 1PN order

The standard form (Misner, Thorne, & Wheeler 1973) of a 1PN metric is the following

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j\,, \quad (1)$$

where the coordinates $x^0 = ct = \epsilon^{-1}t$, x^a ($a = 1,2,3$) all have the dimensions of a length and

$$g_{00} = -1 + 2\epsilon^2 V - 2\epsilon^4 V^2 + O(6)\,,$$

$$
\begin{aligned}
g_{0i} &= -4\epsilon^3 V_i + O(5)\,,\\
g_{ij} &= \delta_{ij}\left(1+2\epsilon^2 V\right) + O(4)\,,
\end{aligned}
\tag{2}
$$

with $\epsilon = 1/c$. The functions V and V_i denote the potentials associated with the gravitational field. For instance, for the spacetime of a single body with mass $\mathcal{M}$ in harmonic coordinates we have

$$
\begin{aligned}
V &= \frac{G\mathcal{M}}{r} + \epsilon^2 G\mathcal{M}\left(-\frac{(n\cdot v)^2}{2r} + 2\frac{v^2}{r}\right),\\
V_i &= \frac{G\mathcal{M}v^i}{r}\,,
\end{aligned}
\tag{3}
$$

whereas for a system of two bodies with masses m_1 and m_2 the potentials are (Blanchet, Faye, & Ponsot 1998)

$$
\begin{aligned}
V &= \frac{Gm_1}{r_1} + \epsilon^2 Gm_1\left[-\frac{(n_1\cdot v_1)^2}{2r_1} + 2\frac{v_1^2}{r_1}\right.\\
&\quad \left.+Gm_2\left(-\frac{r_1}{4r_{12}^3} - \frac{5}{4r_1 r_{12}} + \frac{r_2^2}{4r_1 r_{12}^3}\right)\right]\\
&\quad +1\leftrightarrow 2\,,\\
V_i &= \frac{Gm_1 v_1^i}{r_1} + 1\leftrightarrow 2\,,
\end{aligned}
\tag{4}
$$

and the notation $\mathbf{n}_a = \mathbf{r}_a/r_a$, where

$$
\begin{aligned}
r_a &= r_a(t,x,y,z) = \left[(x-x_a(t))^2 + (y-y_a(t))^2\right.\\
&\quad \left.+(z-z_a(t))^2\right]^{1/2}\,,
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
r_{ab}(t) &= \left[(x_a(t)-x_b(t))^2 + (y_a(t)-y_b(t))^2\right.\\
&\quad \left.+(z_a(t)-z_b(t))^2\right]^{1/2}\,,
\end{aligned}
\tag{6}
$$

has been used, with $a, b = 1, 2$; in the case of the spacetime of a single body (3) r_1 has been simply denoted by r and m_1 by $\mathcal{M}$.

2.1. Fiducial observers and adapted frame

It is convenient to adopt the so-called "threading" point of view (Jantzen, Carini, & Bini 1992), i.e. to select as fiducial observers the static observers with associated 4-velocity $m = (1/M)\partial_0$, i.e. those whose world lines correspond to constant values of the spatial coordinates $x^a = const$. The spacetime metric can thus be re-written as

$$
g = -M^2(dx^0 - M_a dx^a)^2 + \gamma_{ab}dx^a dx^b\,, \tag{7}
$$

with inverse

$$
g^{-1} = -M^{-2}\partial_0\otimes\partial_0 + \gamma^{ab}\epsilon_a\otimes\epsilon_b\,, \tag{8}
$$

where we have introduced the lapse-shift notation

$$
g_{00} = -M^2\,,\quad g_{0a} = M^2 M_a\,. \tag{9}
$$

The triad

$$
\epsilon_a = \partial_a + M_a\partial_0 \tag{10}
$$

defines an observer-adapted (non-orthogonal) spatial frame such that

$$
\begin{aligned}
m\cdot m &= -1\,,\quad m\cdot\epsilon_a = 0\,,\\
\epsilon_a\cdot\epsilon_b &= \gamma_{ab} = g_{ab} + M^2 M_a M_b\,.
\end{aligned}
\tag{11}
$$

Projection onto the Local Rest Space of the observers m is accomplished by means of the operator

$$
P(m) = g + m\otimes m\,. \tag{12}
$$

Furthermore, the cross product between spatial vectors, say X and Y, can be defined there as follows

$$
[X\times_m Y]_a = \eta(m)_{abc}X^bY^c\,, \tag{13}
$$

where $\eta(m)_{abc} = \eta_{\rho abc}m^\rho$.

Together with the (natural) non-orthogonal triad ϵ_a one may form an orthonormal triad adapted to m, i.e.,

$$
\begin{aligned}
e(m)_{\hat 1} &= \frac{1}{\sqrt{\gamma_{11}}}\epsilon_1\,,\\
e(m)_{\hat 2} &= \sqrt{\frac{\gamma_{11}}{\gamma\gamma^{33}}}\left(\epsilon_2 - \frac{\gamma_{12}}{\gamma_{11}}\epsilon_1\right),\\
e(m)_{\hat 3} &= e(m)_{\hat 1}\times_m e(m)_{\hat 2}\\
&= \frac{1}{\sqrt{\gamma\gamma^{33}}}\epsilon_1\times_m\epsilon_2\\
&= \frac{1}{\sqrt{\gamma^{33}}}[\gamma^{31}\epsilon_1 + \gamma^{32}\epsilon_2 + \gamma^{33}\epsilon_3]\,,
\end{aligned}
\tag{14}
$$

where

$$\gamma\gamma^{13} = \gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}\,,$$
$$\gamma\gamma^{23} = \gamma_{13}\gamma_{12} - \gamma_{11}\gamma_{23}\,,$$
$$\gamma\gamma^{33} = \gamma_{11}\gamma_{22} - \gamma_{12}^2\,, \qquad (15)$$

and

$$\gamma = \det[\gamma_{ab}] = \gamma_{11}\gamma_{22}\gamma_{33} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2 + 2\gamma_{12}\gamma_{13}\gamma_{23}\,. \qquad (16)$$

The series expansion of the lapse and shift functions M and M_a and the spatial metric γ_{ab} gives

$$M = M^{(0)} + \epsilon^2 M^{(2)} + \epsilon^4 M^{(4)} + O(6)\,,$$
$$M_a = \epsilon^3 M_a^{(3)} + \epsilon^5 M_a^{(5)} + O(7)\,,$$
$$\gamma_{ab} = \gamma_{ab}^{(0)} + \epsilon^2\gamma_{ab}^{(2)} + O(6)\,, \qquad (17)$$

with

$$M^{(0)} = 1\,, \quad M^{(2)} = -V\,, \quad M^{(4)} = \frac{1}{2}V^2\,,$$
$$M_a^{(3)} = -4V_a\,, \quad M_a^{(5)} = -8VV_a\,,$$
$$\gamma_{ab}^{(0)} = \delta_{ab}\,, \quad \gamma_{ab}^{(2)} = 2V\delta_{ab}\,. \qquad (18)$$

To the 1PN order we then have

$$m = \left(1 + \epsilon^2 V + \epsilon^4\frac{1}{2}V^2\right)\partial_0\,,$$
$$\epsilon_a = \partial_a - 4\epsilon^3 V_a\partial_0\,,$$
$$e(m)_{\hat a} = -4\epsilon^3 V_a\partial_0 + \left(1 - \epsilon^2 V + \epsilon^4\frac{3}{2}V^2\right)\partial_a\,, \qquad (19)$$

also implying

$$P(m)_{ab} = (1 + 2\epsilon^2 V)\delta_{ab} \qquad (20)$$

for the projection operator.

3. Test particle motion

Consider the motion of a test particle in this gravitational background. Let its timelike world line be given by

$$U = \gamma(U,m)[m + \epsilon\nu(U,m)] = \Gamma[\partial_0 + \epsilon v^a\partial_a]\,, \qquad (21)$$

where

$$\nu(U,m) = ||\nu(U,m)||\hat\nu(U,m) = \nu(U,m)^{\hat a} e(m)_{\hat a}\,,$$
$$\gamma(U,m) = (1 - \epsilon^2||\nu(U,m)||^2)^{-1/2}\,, \qquad (22)$$

so that $U\cdot U = -1$ and v^a depend on t only. Here we have included two representations of U, one in terms of the m observers (fundamental for a 1 + 3 observer adapted analysis of the motion) and the other in terms of coordinates (more familiar when using the PN approximation).

An adapted frame to this world line can be obtained by boosting the orthonormal threading frame $\{m, e(m)_{\hat a}\}$ along U, i.e.,

$$E_{\hat 0} = U\,, \qquad E_{\hat a} = B(U,m)e(m)_{\hat a}\,, \qquad (23)$$

where

$$E_{\hat a} = \left[P(U) - \frac{\gamma(U,m)}{\gamma(U,m)+1}\epsilon^2\nu(m,U)\otimes\nu(m,U)\right]\llcorner\, e(m)_{\hat a}\,, \qquad (24)$$

where the symbol $\llcorner$ denotes right contraction and

$$\epsilon\nu(m,U) = \frac{1}{\gamma(U,m)}m - U\,. \qquad (25)$$

At the 1PN level the particle 4-velocity (21) has the Γ factor

$$\Gamma = \Gamma^{(0)} + \epsilon^2\Gamma^{(2)} + \epsilon^4\Gamma^{(4)} = 1 + \epsilon^2\left(\frac{1}{2}v^2 + V\right) + \epsilon^4\left[\frac{3}{8}v^4 - 4\delta^{ab}V_a v_b + \frac{1}{2}V^2 + \frac{5}{2}v^2 V\right]\,, \qquad (26)$$

so that

$$= \Gamma[\partial_0 + \epsilon\mathbf{v}] \qquad (27)$$

where the notation

$$\mathbf{v} = v^a\partial_a\,, \qquad v^2 = \delta_{ab}v^a v^b \qquad (28)$$

has been used. In terms of the coordinate components of the spatial velocity the frame components $\nu(U,m)^{\hat a}$ and the associated Lorentz factor $\gamma(U,m)$ are

$$\nu(U,m)^{\hat a} = (1 + 2V\epsilon^2)v^a\,,$$
$$\gamma(U,m) = 1 + \frac{v^2}{2}\epsilon^2 + \left(2V + \frac{3}{8}v^2\right)v^2\epsilon^4\,. \qquad (29)$$

The spatial triad (24) adapted to U is thus given by

$$E_{\hat a} = \left\{\epsilon v^a + \epsilon^3\left[v^a\left(\frac{v^2}{2}+3V\right)-4V_a\right]\right\}\partial_0 + \left(1-\epsilon^2 V+\frac{3}{2}\epsilon^4 V^2\right)\partial_a + v^a\frac{\epsilon^2}{2}\left[1+3\epsilon^2\left(\frac{v^2}{4}+V\right)\right]\mathbf{v}\,. \tag{30}$$

If the particle moves along a timelike geodesic, its motion is governed by the equations

$$\begin{aligned} a^a \equiv \frac{dv^a}{dt} &= [\nabla V]^a \\ &+\epsilon^2\left[4(\partial_t\mathbf{V})^a - 3v^a\partial_t V - 4(\mathbf{v}\times\mathbf{V})^a \right.\\ &\left.+(v^2-4V)(\nabla V)^a - 4v^a(\mathbf{v}\cdot\nabla V)\right] \\ &+\epsilon^4[6v^a(v^2-3V)\partial_t V + 4(V-v^2)(\partial_t\mathbf{V})^a \\ &-4v^a(\mathbf{v}\cdot\partial_t\mathbf{V})]\,, \end{aligned} \tag{31}$$

where $[\nabla V]^a = \delta^{ab}\partial_b V$.

3.1. Fermi-Walker frame along the particle's world line

One may also require that the fiducial triad adapted to U be Fermi-Walker transported along U, a fact that might be given an operational definition, since Fermi-Walker transported axes are obtained by gyroscope-fixed axes. An orthonormal Fermi-Walker transported triad $F_{\hat a}$ can be constructed after a suitable rotation of the orthonormal frame $E_{\hat a}$. However, even if – in principle – explicit calculations can always be performed, in any specific fixed background spacetime may arise computational difficulties; the latter disappear when dealing with PN approximations.

The spatial triad (30) rotates with a certain angular velocity $\omega_{\rm (fw)}$ with respect to gyro-fixed axes along U, i.e.,

$$P(U)\nabla_U E_{\hat a} \equiv \nabla_{\rm (fw)}(U)E_{\hat a} = \omega_{\rm (fw)}\times E_{\hat a}\,, \tag{32}$$

with

$$\omega_{\rm (fw)} = \epsilon^3 X^a E_{\hat a}\,, \tag{33}$$

where

$$\mathbf{X} = \frac{1}{2}(\mathbf{v}\times\mathbf{a}) - 2\,\mathrm{curl}\,(\mathbf{V}-\mathbf{v}V)\,, \tag{34}$$

because v^a only depend on t; here the $\times$ and curl operations are defined as in the case of flat 3-dimensional Euclidean space.

In order to obtain a Fermi-Walker transported orthonormal frame $\{U, F_{\hat a}\}$ the spatial triad $\{E_{\hat a}\}$ must be conveniently rotated, namely

$$F_{\hat a} = \mathcal{R}^{\hat b}{}_{\hat a}E_{\hat b}\,, \tag{35}$$

where $\mathcal{R}$ is the rotation matrix such that

$$\delta_{\hat a\hat b} = \mathcal{R}^{\hat d}{}_{\hat a}\delta_{\hat d\hat c}\mathcal{R}^{\hat c}{}_{\hat b}\,. \tag{36}$$

The Fermi-Walker transport condition $P(U)\nabla_U F_{\hat a} = 0$ then implies along the particle's world line (i.e., $x^\alpha = x^\alpha(\tau)$)

$$\nabla_U(\mathcal{R}^{\hat b}{}_{\hat a})E_{\hat b} + \mathcal{R}^{\hat b}{}_{\hat a}\omega^{\hat c}_{\rm (fw)}(E_{\hat c}\times E_{\hat b}) = 0\,, \tag{37}$$

that is

$$\nabla_U(\mathcal{R}^{\hat d}{}_{\hat a}) + \mathcal{R}^{\hat b}{}_{\hat a}\omega^{\hat c}_{\rm (fw)}\epsilon_{\hat c\hat b}{}^{\hat d} = 0\,. \tag{38}$$

However, the derivative along the particle's world line may be re-expressed in terms of derivative with respect to the coordinate time according to the relation

$$\nabla_U \quad\to\quad \Gamma\partial_0 = \epsilon\Gamma\partial_t\,, \tag{39}$$

so that Eq. (38) becomes

$$\epsilon\Gamma\partial_t(\mathcal{R}^{\hat d}{}_{\hat a}) = -\mathcal{R}^{\hat b}{}_{\hat a}\epsilon^3 X^{\hat c}\epsilon_{\hat c\hat b}{}^{\hat d}\,, \tag{40}$$

which should be solved order by order. Therefore, taking also a series expansion of $\mathcal{R}^{\hat d}{}_{\hat a}$ it follows that

$$\mathcal{R}^{\hat d}{}_{\hat a} = \delta^{\hat d}{}_{\hat a} + \epsilon^2[\mathcal{R}^{(2)}]^{\hat d}{}_{\hat a} + \epsilon^4[\mathcal{R}^{(4)}]^{\hat d}{}_{\hat a}\,. \tag{41}$$

The orthonormality condition (36) at the order $O(4)$ implies that $[\mathcal{R}^{(2)}]$ is antisymmetric and the symmetric part of $[\mathcal{R}^{(4)}]$ satisfying the condition

$$2[\mathcal{R}^{(4)}]_{(\hat a\hat b)} = [\mathcal{R}^{(2)}]^2{}_{\hat a\hat b}\,. \tag{42}$$

It is easy to check that $[\mathcal{R}^{(2)}]^{\hat{d}}{}_{\hat{a}} = \epsilon^{\hat{d}}{}_{\hat{a}\hat{c}} Y^{\hat{c}}$, namely

$$[\mathcal{R}^{(2)}]^{\hat{d}}{}_{\hat{a}} = \begin{bmatrix} 0 & Y^{\hat{3}} & -Y^{\hat{2}} \\ -Y^{\hat{3}} & 0 & Y^{\hat{1}} \\ Y^{\hat{2}} & -Y^{\hat{1}} & 0 \end{bmatrix}, \qquad (43)$$

where $Y^{\hat{c}} = \int X^{\hat{c}} dt$. From condition (42) we have that the symmetric part of $[\mathcal{R}^{(4)}]$ is known in terms of $[\mathcal{R}^{(2)}]$; hence we may write

$$[\mathcal{R}^{(4)}]_{\hat{a}\hat{b}} = \frac{1}{2}[\mathcal{R}^{(2)}]^2{}_{\hat{a}\hat{b}} + [W^{(4)}]_{\hat{a}\hat{b}} \qquad (44)$$

with $[W^{(4)}]_{\hat{a}\hat{b}}$ antisymmetric. Since the are no evolution equations at order 4, the matrix W remains arbitrary at this order and can be set as identically vanishing. A Fermi-Walker spatial triad at the order $O(4)$ turns out to be then given by

$$F_{\hat{a}} = E_{\hat{b}} \left[I - \epsilon^2 [\mathcal{R}^{(2)}] + \frac{\epsilon^4}{2} [\mathcal{R}^{(2)}]^2 \right]^{\hat{b}}{}_{\hat{a}}, \qquad (45)$$

that is

$$F_{\hat{a}} = E_{\hat{b}} \left[\exp[-\epsilon^2 [\mathcal{R}^{(2)}]] \right]^{\hat{b}}{}_{\hat{a}}, \qquad (46)$$

where I is the identity matrix. Note that the Fermi-Walker frame (45) reduces to that obtained by Fukushima (1988) if truncated to the third order in the expansion (modulo a misprint in the frame components of the Fermi-Walker angular velocity (33)).

4. Conclusions

We have provided the mathematical formalism necessary to explicitly construct a Fermi-Walker transported frame along an accelerated timelike world line of a generic 1PN approximated metric. If the considered path is geodesic, the frame is parallely transported. In view of the relative easiness in technically realize such a frame with mutually orthogonal gyroscopes, the result of the present analysis may play a role both in astrometric contexts and satellite navigation in space.

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