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Physics and coordinates in competition in highly accurate measurements

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Abstract. Advancement in astronomical observations requires coding light propagation at high level of precision; this could open a new detection window of many subtle relativistic effects suffered by light while it is propagating and recorded in the physical measurements. Light propagation and its subsequent detection should indeed be conceived in a fully relativistic context, in order to interpret the results of the observations in accordance with the geometrical environment affecting light propagation itself. This contribution aims to bring the attention on some physical aspects of the problem that guarantees consistency of the measured physical effects to the intrinsic accuracy of the space-time.

1. Introduction

The treatment of light propagation in time-dependent gravitational fields, the fabric of our surrounding universe, is extremely important for astrophysics, encompassing issues from fundamental astronomy to cosmology (Will (2006), Turyshev (2008), Crosta and Mignard (2006), Kopeikin and Gwinn (2000), Damour and Nordtvedt (1993), Uzan (2010), de Felice et al., (2011) and references therein). As matter of fact, the trajectory of a photon is traced by solving the null geodesic in a curved space-time dictated by General Relativity (GR), a theory in which geometry and physics are joined together. At the same time, the detection process usually takes place in a geometrical environment generated by a n-body distribution as it is that of our Solar System. Nowadays, a few approaches exist that model light propagation in a relativistic context. Among them, the post-Newtonian (pN) and the post-Minkowskian (pM) approximations are those mainly used (Kopeikin and Mashhoon (2002); Klioner (2003); Teyssandier and Le Poncin-Lafitte (2008) and references therein). In this context, an alternative method is represented by RAMOD (de Felice et al. (2004), 2006), a family of astrometric models of increasing intrinsic accuracy conceived to solve the inverse ray-tracing problem in a general relativistic framework, according to the precepts of measurement in GR. Therefore, it uses a 3+1 characterization of space-time in order to measure physical phenomena along the proper time and on the rest-space of a set of fiducial observers (de Felice and Bini 2010). This contribution intends to bring the attention toward the physical consistency that one needs to keep in order to solve the light tracing problem in situation of increasingly accurate detections. In fact, one should scrutinize if the measurement is local or not with respect to the curvature generated by the gravitational field where observations take place, as it is discussed in this volume by Bini and de Felice.

2. The astrometric problem

The astrometric problem consists, firstly, in solving the null geodesic for the single stellar photon, in order to trace back the light trajectory to the initial position of the emitting source and, then, determine its astrometric parameters through the astrometric observable, according to the chosen reference frames. Differently from the other approaches, RAMOD's full solution requires the integration of a set of coupled non-linear differential equations, called "master equations". The unknown of these equations is the local line-of-sight $\bar{\boldsymbol{\ell}}$ as measured by the fiducial observer $\mathbf{u}$ at the point of observation in her/his rest-space. At the time of observation, $\bar{\boldsymbol{\ell}}$ provides the boundary condition for uniquely solving the light path by means of the relativistic definition of the observable (Crosta and Vecchiato 2010) and the satellite-observer frames (Bini et al. 2003). The main purpose of the RAMOD approach is to express the null geodesic through all the physical quantities entering the process of measurement without any approximations, in order to entangle all the possible interactions of light with the background geometry. Solving the astrometric problem in practice means to compile an astrometric catalog with the same order of accuracy as the measurements. To what extent, then, is the process of star coordinate reconstruction consistent with General Relativity &Theory of Measurements?

3. With or without vorticity

Gaia-like measurement takes place inside the Solar System, i.e. a weakly relativistic gravitationally bound system, described by the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2)$. Now, in order to gauge how much curvature can be considered local or not with respect to the measurement, let us resort the virial theorem which requires an energy balance of the order of $|h_{\alpha\beta}| \leq U/c^2 \sim v^2/c^2$, where v is the characteristic relative velocity within the system [1]. Therefore the level of accuracy is fixed by the order of the small quantity $\epsilon \sim (v/c)$. Since the system is weakly relativistic, the perturbation tensor $h_{\alpha\beta}$ contributes with even terms in ϵ to g_{00} and g_{ij} (lowest order ϵ^2) and with odd terms in ϵ to g_{0i} (lowest order ϵ^3, Misner et al. 1973; de Felice and Bini 2010); its spatial variations are of the order of $|h_{\alpha\beta}|$, while its time variation is of the order of $\epsilon|h_{\alpha\beta}|$. This means that at the order of ϵ^3, not only the time dependence of the background metric cannot be ignored any longer, but also the vorticity, which measures - in the process of foliation- how a world-line of an observer rotates around a neighboring one, can be neglected being proportional to the g_{0i} term of the metric (see details in Crosta 2011). Consequently, it is not possible to define a rest-space of a fiducial observer that covers the entire space-time. Any observer $\mathbf{u}$ can be considered at rest with respect to the coordinates x^i *only locally*, and for this reason $\mathbf{u}$ is called the *local barycentric observer*, as identified in de Felice et al. (2006). The master equations satisfied by the vector field $\bar{\boldsymbol{\ell}}$ up to the ϵ^3 order of accuracy are

$$\frac{d\bar{\ell}^0}{d\sigma} = \bar{\ell}^i\bar{\ell}^j h_{0j,i} + \frac{1}{2}h_{00,0} \tag{1}$$

$$\begin{aligned}\frac{d\bar{\ell}^k}{d\sigma} &= \frac{1}{2}\bar{\ell}^k\bar{\ell}^i\bar{\ell}^j h_{ij,0} - \bar{\ell}^i\bar{\ell}^j\left(h_{kj,i} - \frac{1}{2}h_{ij,k}\right) \\ &- \frac{1}{2}\bar{\ell}^k\bar{\ell}^i h_{00,i} - \bar{\ell}^i\left(h_{k0,i} + h_{ki,0} - h_{0i,k}\right) \\ &+ \frac{1}{2}h_{00,k} + \bar{\ell}^k\bar{\ell}^i h_{0i,0} - h_{k0,0},\end{aligned} \tag{2}$$

named "RAMOD4 master equations" in the dynamical case (de Felice et al. 2006; Crosta 2011), being σ the parameter of the null geodesic. Note that there is a differential equation also for the $\bar{\ell}^0$ component, which represents an opportunity to better decipher light propagation in future developments.

The ϵ^2 regime, instead, is referred as the "static case", or "static space-time", i.e. a stationary space-time in which a time-like Killing vector field $\mathbf{u}$ has vanishing vorticity (de Felice et al. 2004). In this case the parameter σ on u is the proper time of the physical observers who transport the spatial coordinates without shift. Any hypersurface $t(x, y, z) = constant$, at

[1] For a typical velocity $\sim$ 30 km/s, $(v/c)^2 \sim 1$ milli-arcsec

each different coordinate time t, can be considered the rest space *everywhere* of the observer $\mathbf{u}$ and the geometry that each photon feels is, then, identified with the weak relativistic metric where $g_{0i} = 0$. In these circumstances we can define a one-parameter local diffeomorphism which maps each point of the null geodesic to the point on the slice at the time of observation, say $S(t_o)$ (de Felice et al. 2004):

$$\frac{d\bar{\ell}^k}{d\sigma} = -\bar{\ell}^k\left(\frac{1}{2}\bar{\ell}^i h_{00,i}\right) - \delta^{ks}\left(h_{sj,i} - \frac{1}{2}h_{ij,s}\right)\bar{\ell}^i\bar{\ell}^j + \frac{1}{2}\delta^{ks}h_{00,s}. \quad (3)$$

Equations (3) determine light propagation in the static case, and are called "RAMOD3 master equations" (de Felice et al. 2004). Only a vorticity-free space-time allows to parametrize simultaneously the mapped trajectory with respect to the Center of Mass on $S(t_o)$; if the Euclidean scalar product is applied, the RAMOD procedure for the parametrization generalizes the one used in Kopeikin and Mashhoon (2002) or Klioner (2003) (Crosta 2011).

The fact that light tracing is different with or without the vorticity term make evident how the RAMOD recipe, based on a measurement protocol, differs from a direct "coordinate" approach which, instead, does not discriminate the accuracy of the geometry to be involved.

4. Physics and coordinates: matching the interpretation at high accuracy

The quantity $\bar{\ell}$ is the unitary four-vector representing the *local line-of-sight* of the incoming photon as measured by the local observer $\mathbf{u}$ in his/her gravitational environment; it represents a physical quantity in any case, with or without vorticity. By implementing its coordinate expressions straightway, equations (2), i.e. those for the spatial components, are converted into the coordinate ones derived in Kopeikin and Mashhoon (2002) at the first pM approximation of the null geodesic (Crosta 2011). This result was expected, since both models are deduced from the null geodesic in a weak field regime. Then, once such an equivalence is obtained, one could solve the master equation in the RAMOD framework by applying the same procedure adopted in Kopeikin and Mashhoon (2002). However, consistently with the reasoning of the previous section, only RAMOD3 master equation can be transformed into the solution given by Kopeikin and Mashhoon (2002), since the parametrization in RAMOD is possible only in a vorticity-free space-time, i.e. at the ϵ^2 level of accuracy in the h linear regime. In fact, if one assumes a constant light direction and a perturbed straight line trajectory, the equivalence of the two parametrizations implies a change of coordinates which transforms equation (2) into the same parametrized equation (36) used in Kopeikin and Mashhoon (2002), obtained, instead, by plugging the g_{0i} term of the metric directly into the geodesic equation without discriminating the accuracy of the involved background geometry. Nevertheless, the integration of the null geodesic in Kopeikin and Mashhoon (2002) intends to consider the gravitomagnetic effects. In addition, the metric coefficients $h_{\alpha\beta}$ depend on the retarded distance $r_{(a)}$ as discussed in de Felice et al. (2006). This means that one has to compute the spatial coordinate distance $r_{(a)}$ from the points on the photon trajectory to the a-th gravity source at the appropriate retarded time and up to the required accuracy. Hence, if we wish our model be accurate to ϵ^3, it suffices that the retarded distance r contributes to the gravitational potentials, which we remind are at the lowest of order ϵ^2, with terms of the order of ϵ. Instead, to the order of ϵ^2 (static geometry), the contribution of the relative velocities of the gravitating sources can be neglected. Indeed, in the static case one can choose to further expand the retarded distance in order to keep the terms depending on the source's velocity up to the desired accuracy. Obviously the gravitational field does not vary, since the terms g_{0i} are null and time derivatives of the metric are at lowest of order ϵ^3; therefore, the effects due to the bodies' velocity cannot be related to a dynamical change of space-time, at least up to the scale where the vorticity can be neglected. Actually, the positions of the bodies can be recorded as subsequent snapshots onto the mapped trajectories and deduced as ”postponed” corrections in the reconstruction of the photon’s path.

The importance of the measurement protocol in setting the correct role of the coordinates, and thus avoiding misinterpretations of parallel but different quantities, is also discussed in Crosta and Vecchiato (2010), where, within the context of the Gaia mission (ESA, Turon et al. 2005), a first comparison between RAMOD and GREM (Gaia RElativistic Model, Klioner 2003) was carried out via the extrapolation of the aberrational term in the *local* light direction. Differences, that already exist at the level of the aberration effect, suggest particular care in the interpretation of the final catalog. Another example which shows how the accurate inclusion of the geometry redraws a standard measurement, is given by the formula for the Doppler shift in de Felice et al., (2011). The spectroscopic and astrometric data that will be provided by the new generation of satellites can be implemented with one another, thus leading to a general-relativistic Doppler which is exact up to and including the ϵ^3 terms. It is also showed that a previously proposed Doppler-shift formula is definitely not adequate to this task, since it misses relevant relativistic corrections already at ϵ^2.

5. Conclusions

Modeling light propagation is intrinsically connected to the identification of the geometry where photons naturally move. The different conception of RAMOD provides a method to exploit high accurate observations to their full extent, as it could be the case for the astrometric data coming from the ESA mission Gaia, possibly a new beginning in the field of relativistic astrometry. In RAMOD the vorticity term cannot be neglected at the order of ϵ^3: ignoring it locally is valid only in a small neighborhood compared to the scale of vorticity itself. When the vorticity term is needed the light trajectory cannot be laid out on a unique rest-space of simultaneity from the observer to the star, wherever the latter could be located. Without vorticity RAMOD allows a parametrization of the light trajectory and sets the level of reciprocal consistency with the existing approaches. Only RAMOD4, i.e. the case of a dynamical space-time, fully preserves the active content of gravity. Its master equations are not contemplated in other approaches and its solution could be the clue for uncovering new effects. The local line-of-sight, as a physical entity, can be used in the future for an inverse parameter problem approach, able to statistically determine the metric also outside the Solar System (Tarantola 2005).

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